Problem 1
Use contour integration to evaluate the integral
\[ \int_{0}^{2\pi} \frac{d\theta}{5 - 4 \sin \theta} \]

Problem 2
Let \( x^1, x^2, x^3 \) be Cartesian coordinates on a Euclidean three-dimensional space \( M = \mathbb{R}^3 \) and let \( \mathcal{X}(M) \) denote the space of smooth vector fields on it. For any vector field \( X \) define one- and two-forms \( \omega^1_X \) and \( \omega^2_X \) by
\[ \omega^1_X(Y) = (X, Y), \quad \omega^2_X(Y, Z) = (X, Y, Z). \]
Here \((X, Y)\) is the dot product of \( X \) and \( Y \), and \((X, Y, Z)\) is the triple product of \( X, Y, \) and \( Z \) (which equals to the determinant of the matrix with columns \( X, Y, \) and \( Z \)).

Consider operations \( curl : \mathcal{X}(M) \to \mathcal{X}(M) \), and \( div : \mathcal{X}(M) \to C^\infty(M) \) defined by the formulas
\[ d\omega^1_Y = \omega^2_{curl Y}, \quad d\omega^2_Z = (div Z)dx^1 \wedge dx^2 \wedge dx^3. \]

a) Demonstrate that the exterior product of one-forms corresponds to the cross product of vector fields, by proving \( \omega^1_X \wedge \omega^1_Y = \omega^2_{X \times Y} \).

b) Demonstrate that the exterior product of a one-form and a two-form corresponds to the dot product of vector fields: \( \omega^1_X \wedge \omega^2_Y = (X, Y)dx^1 \wedge dx^2 \wedge dx^3 \).

c) Express the following relation in terms of differential forms:
\[ div(X \times Y) = (curl X, Y) - (curl Y, X). \]

d) Prove this relation using its differential form presentation.
Problem 3
Consider the following map
\[
F : \mathbb{R}^3 \to \mathbb{R}^3 \\
\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x(x - y + z) - yz \\ x^2 + y^2 - 2xy \\ y - z \end{pmatrix}.
\]
Let \( C = \{(u, v, w) \in \mathbb{R}^3 | u^2 + v^2 = 1, w = 0\} \) denote a unit circle in \( \mathbb{R}^3 \).
Is its preimage \( F^{-1}(C) \) of \( C \) a regular submanifold?
If it is, what is its dimension? If it is not, why not?

Problem 4
Compute the fundamental group of \( \mathbb{R}^3 \) with two lines removed. To receive full credit, you should consider different cases.

Problem 5
Let \( \pi : S^2 \to \mathbb{RP}^2 \) be the natural projection, sending a point \((x, y, z) \in S^2\) to the point \((x : y : z) \in \mathbb{RP}^2\). Show that the map \( \pi \) does not admit a continuous right inverse, i.e. there exists no continuous map \( \phi : \mathbb{RP}^2 \to S^2 \) such that \( \pi \circ \phi = \text{id} \).

Problem 6
Let \( M \) be an \( n \)-dimensional manifold, and let \( \phi : M \to M \) be a continuous map such that the induced map \( \phi_* : H_n(M, \mathbb{Z}) \to H_n(M, \mathbb{Z}) \) is given by \( \phi_*(\xi) = -\xi \). Define the space \( X \) as the quotient of \( M \times [0, 1] \) by the following equivalence relation: \( (x, 0) \sim (\phi(x), 1) \).

1. Let \( \pi : M \times [0, 1] \to X \) be the quotient map. Define the set \( A \) as an open neighborhood of \( X_1 = \pi(M \times [0, 1/2]) \) which deformation retracts to \( X_1 \). Similarly, let \( B \) be an open neighborhood of \( X_2 = \pi(M \times [1/2, 1]) \) which deformation retracts to \( X_2 \). Describe the map
\[
H_n(A \cap B, \mathbb{Z}) \to H_n(A, \mathbb{Z}) \oplus H_n(B, \mathbb{Z})
\]
in the Mayer-Vietoris sequence for the decomposition \( X = A \cup B \).

2. Hence show that \( H_{n+1}(X, \mathbb{Z}) = 0 \).