Existence & Uniqueness of Solutions via Picard Iteration

1. Repeat the steps in Example 2 with the equation \( y'(t) = -y \), with \( y_0 = 1 \).
2. Repeat the steps in Example 2 with the equation \( y'(t) = -y \), with \( y_0 = 1 \).
3. Discuss whether you could repeat the steps in Example 2 with the equation \( ty'(t) = y - 1 \).
4. Discuss whether you could repeat the steps in Example 2 with the equation \( y'(t) = y^{1/3} + t \).
Systems of Linear Equations

5. Express $e^{At}$ by its Taylor Series to verify that $y = e^{At}y_0$ satisfies the differential equation $y' = Ay$ subject to $y = y_0$ at $t = 0$. The Taylor Series converges for all $R \in \mathbb{R}^2$, so you may differentiate term-by-term.

6. Compute $e^A$ for
   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$
   (b) $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

7. Sketch phase planes for the following systems of differential equations
   (a) $y' = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} y = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^T \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} t & 1 \\ -1 & 1 \end{bmatrix}^{-1} y$
   (b) $y' = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} y = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} y$
   (c) $y' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} y = \begin{bmatrix} i & -i \\ 1 & -1 \end{bmatrix}^T \begin{bmatrix} 1 + 2i & 0 \\ 0 & 1 - 2i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & -1 \end{bmatrix}^{-1} y$

8. Sketch the phase plane for the following system of differential equations
   \[
   \begin{align*}
   \dot{x}(t) &= -y(t) \\
   \dot{y}(t) &= x(t) \\
   \dot{z}(t) &= -z(t)
   \end{align*}
   \]
Techniques for Explicit Solutions

9. Find the general solution to
\[ \frac{dP}{dt} = kP \left( 1 - \frac{P}{K} \right) \] (1)

10. Find the general solutions to the following differential equations
   
   (a) \( y'' + 3y' - 10y = 0 \)
   (b) \( \ddot{x} + 4\dot{x} + 4x = 0 \)

11. Find the general solutions to the following differential equations

   (a) \( s^2y'' - 2sy' + 2y = 0 \)
   (b) \( t^2\ddot{x} + 3t\dot{x} + 2x = 0 \)

12. Solve
\[ s \frac{dy}{ds} + (s + 1)y = 3 \] (2)

13. Use the method of the variation of parameters so solve the following differential equations

   (a) \( s^2y'' - 2sy' + 2y = s \)
   (b) \( \ddot{x} + 4\dot{x} + 4x = \cosh t \)