Integration Workshop 2020
– Ordinary Differential Equations –
Problems

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Existence & Uniqueness of Solutions via Picard Iteration

1. Repeat the steps in Example 2 with the equation $y'(t) = -y$, with $y_0 = 1$.
2. Repeat the steps in Example 2 with the equation $y'(t) = -y$, with $y_0 = 1$.
3. Discuss whether you could repeat the steps in Example 2 with the equation $ty'(t) = y - 1$.
4. Discuss whether you could repeat the steps in Example 2 with the equation $y'(t) = y^{1/3} + t$.

Systems of Linear Equations

5. Express e^{At} by its Taylor Series to verify that $\boldsymbol{y} = e^{At}\boldsymbol{y_0}$ satisfies the differential equation $\boldsymbol{y}' = A\boldsymbol{y}$ subject to $\boldsymbol{y} = \boldsymbol{y}_0$ at $t = 0$. The Taylor Series converges for all $\boldsymbol{R} \in \mathbb{R}^2$, so you may differentiate term-by-term.

6. Compute e^A for

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$

7. Sketch phase planes for the following systems of differential equations

   (a) $\boldsymbol{y}' = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \boldsymbol{y} \quad \left(= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \boldsymbol{y} \right)$

   (b) $\boldsymbol{y}' = \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \boldsymbol{y} \quad \left(= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \boldsymbol{y} \right)$

   (c) $\boldsymbol{y}' = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \boldsymbol{y} \quad \left(= \begin{bmatrix} i & -i \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix} \begin{bmatrix} i & -i \\ 1 & -1 \end{bmatrix}^{-1} \boldsymbol{y} \right)$

8. Sketch the phase plane for the following system of differential equations

$$\begin{aligned} \dot{x}(t) &= -y(t) \\ \dot{y}(t) &= x(t) \\ \dot{z}(t) &= -z(t) \end{aligned}$$

Techniques for Explicit Solutions

9. Find the general solution to
$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \tag{1}$$

10. Find the the general solutions to the following differential equations
    (a) $y'' + 3y' - 10y = 0$
    (b) $\ddot{x} + 4\dot{x} + 4x = 0$

11. Find the general solutions to the following differential equations
    (a) $s^2y'' - 2s\dot{y} + 2y = 0$
    (b) $t^2\ddot{x} + 3t\dot{x} + 2x = 0$

12. Solve
$$s\frac{dy}{ds} + (s+1)y = 3 \tag{2}$$

13. Use the method of the variation of parameters so solve the following differential equations
    (a) $s^2y'' - 2sy' + 2y = s$
    (b) $\ddot{x} + 4\dot{x} + 4x = \cosh t$