LINEAR ALGEBRA PROBLEM SET
UNIVERSITY OF ARIZONA INTEGRATION WORKSHOP (2019)

1. Lecture 1

Problem 1.1. *Prove that matrix multiplication is associative by relating this operation to composition of linear maps (via choices of bases).*

Problem 1.2. *Let V be a vector space with a basis S. Is S^* always a basis of V^* in general? If not, give a counter-example.*

Problem 1.3. *Let U_1, U_2 be subspaces of a finite dimensional vector space V. Prove that*

$$\dim(U_1 + U_2) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Problem 1.4. *Let $T : V \to W$ be a linear transformation and $T^\vee : W^* \to V^*$ the dual (transpose) map. Prove that $\ker(T) = \operatorname{im}(T^\vee)^\perp$ where for a subspace $U \subset V^*$ we set*

$$U^\perp := \{\mathbf{v} \in V : u(\mathbf{v}) = 0 \text{ for all } u \in U\}$$

Similarly, prove that $\operatorname{im}(T) = \ker(T^\vee)^\perp$.

Problem 1.5. *If W is a subspace of a finite dimensional vector space V with $\dim(W) = \dim(V) - 1$, show that there exists $f \in V^*$ with $W = \ker(f)$.*

Problem 1.6. *Prove Theorem 1.29.*

Problem 1.7. *Prove Theorem 1.30.*

Problem 1.8. *In the statement of Theorem 1.30, can we further conclude $W \oplus W^\perp \cong V$ in general? If not, give a counter-example.*

Problem 1.9. *Let n be a positive integer and $P(x)$ be a degree n polynomial with complex coefficients such that $P(0), P(1), \dots, P(n)$ are all integers. Prove that the polynomial $n!P(x)$ has integer coefficients, using the fact that the polynomials*

$$\binom{x}{m} = \frac{x(x-1)\cdots(x-m+1)}{m!}, \ m = 0, 1, 2, \dots$$

form a basis of a vector space $\mathbf{C}[x]$.

Problem 1.10. *Let V be a vector space over $\mathbf{R}$ and let $f, f_1, \dots, f_n$ be linear maps from V to $\mathbf{R}$. Suppose that $f(x) = 0$ whenever $f_1(x) = f_2(x) = \cdots = f_n(x) = 0$. Prove that f is a linear combination of $f_1, \dots, f_n$.*

Problem 1.11. *We have n coins of unknown masses of some positive real values and a balance. We are allowed to place some of the coins on one side of the balance and an equal number of coins on the other side. After thus distributing the coins, the balance gives a comparison of the total mass of each side, either by indicating that the two masses are equal or by indicating that a particular side* *is the more massive of the two. By connecting the situation with a system of linear equations whose unknowns are the n masses, show that at least $n-1$ such comparisons are required to determine whether all of the coins are of equal mass.*

Problem 1.12. *There are given $2n+1$ real numbers, with the property that whenever one of them is removed, the remaining $2n$ can be split into two sets of n elements that have the same sum of elements. Prove that all the numbers are equal, by connecting the situation with a system of linear equations.*

2. Lecture 2

Problem 2.1. *Let A and B be $n \times n$ matrices. Prove that AB and BA have the same characteristic polynomial.*

Problem 2.2. *The order n Vandermonde determinant is*

$$V_n := \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}$$

Prove that $(x_j - x_i)$ divides V_n for all $i < j$ and conclude that

$$V_n = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Problem 2.3. *Let $P(t)$ be a polynomial of even degree with real coefficients. Prove that the function $f : \mathrm{Mat}_{n\times n}(\mathbf{R}) \to \mathrm{Mat}_{n\times n}(\mathbf{R})$ defined by $f(X) = p(X)$ is not onto, by considering determinant of matrices.*

Problem 2.4. *Let α be a non-zero real number, and let $F, G : \mathbf{R}^n \to \mathbf{R}^n$ be linear maps such that $FG - GF = \alpha F$.*

(1) Show that for all integer $k \ge 1$, we have $F^k G - GF^k = \alpha k F^k$.
(2) Show that $F^k = 0$ for some $k \ge 1$.

Problem 2.5. *Let V be a finite dimensional vector space over K with $\mathrm{char}(K) \neq 2$. A linear map $T : V \to V$ is said to be an* ***involution*** *if $T^2 = \mathrm{Id}$. Prove that for any involution T, V has a basis consisting of eigenvectors of T.*

Problem 2.6. *Let A and B be 2×2 matrices with determinant 1. Using Cayley-Hamilton theorem, prove*

$$\mathrm{tr}(AB) - (\mathrm{tr}(A))(\mathrm{tr}(B)) + \mathrm{tr}(AB^{-1}) = 0.$$

Problem 2.7. *Prove Perron-Frobenius theorem (Theorem 2.13) via following steps.*

(1) Let L be the intersection of $[0,\infty)^n$ with the n-dimensional unit sphere. We want to show that $f : L \to L$ given by $f(\mathbf{v}) = \dfrac{A\mathbf{v}}{||A\mathbf{v}||}$ has a fixed point. The idea is to use Lemma 2.15 with a suitable metric. Let δ be the Hilbert metric on L defined by

$$\delta(\mathbf{v}, \mathbf{w}) = \log\left(\max_i\{\frac{v_i}{w_i}\}/\min_i\{\frac{v_i}{w_i}\}\right)$$

for $\mathbf{v} = (v_1, \dots, v_n)$, $\mathbf{w} = (w_1, \dots, w_n) \in K$. *Check that* δ *satisfies the triangle inequality.*

(2) *Show that* $\delta(f(\mathbf{v}), f(\mathbf{w})) < \delta(\mathbf{v}, \mathbf{w})$ *for any* $\mathbf{v}, \mathbf{w} \in L$ *with* $\mathbf{v} \neq \mathbf{w}$.

(3) *Check that* $f(L)$ *is bounded with respect to the Hilbert norm. Hence, if* L_0 *is the closure of* $f(L)$ *with respect to the Hilbert norm, then* f *is contractive on* L_0. *Its fixed point is the unique eigenvector of* A *with positive entries.*

(4) *Prove the remaining part of the theorem.*

Problem 2.8. *Let* $x_1(t), \dots, x_n(t)$ *be real-valued differentiable functions on* $\mathbf{R}$ *satisfying the system of differential equations*

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n, \ \ i = 1, \dots, n.$$

for some positive real constants $a_{ij} > 0$. *Suppose that* $x_i(t) \to 0$ *as* $t \to \infty$ *for all* $i = 1, \dots, n$. *Prove that* $x_1(t), \dots, x_n(t)$ *are linearly dependent.*

3. Lecture 3

Problem 3.1. *Prove that any invertible real matrix* A *can be written as a product* QP *with* Q *orthogonal and* P *symmetric and positive definite (called the polar decomposition of* A*).*

Problem 3.2. *Give an example of a complex matrix whose associated linear map is normal but neither Hermitian nor unitary.*

Problem 3.3. *Complete the proof of the Spectral theorem (Theorem 3.6).*

Problem 3.4. *Give an example of an inner product space* V *over* $\mathbf{R}$ *and an orthogonal transformation* $T : V \to V$ *such that* V *does not have an orthogonal basis consisting of eigenvectors for* T.

Problem 3.5. *Let* V *be an inner product space over* $\mathbf{R}$ *and* $T : V \to V$ *be an orthogonal transformation. Prove that there exists a basis of* V *such that the matrix of* T *with respect to the basis consists of blocks*

$$\begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & M_r \end{pmatrix}$$

such that each M_i *is either* 1×1 *or* 2×2 *of the following types:*

$$\begin{pmatrix} 1 \end{pmatrix}, \ \begin{pmatrix} -1 \end{pmatrix}, \ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Problem 3.6. *Prove Jordan-Chevalley decomposition (Theorem 3.14) via following steps.*

(1) *Prove the theorem in the special case when the base field* K *is algebraically closed, by considering the associated Jordan canonical form.*

(2) *Consider the general situation when* K *is only assumed to be perfect. Prove the theorem using Galois theory.*

Problem 3.7. *Suppose that the base field* K *is imperfect. Do we still have Jordan-Chevalley decomposition in general? If not, give a counter-example.*