Review of Basic Analysis

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1. Sequences and Series

<u>Def 1.1</u> A set $\mathcal{M}$ and a function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ are called a **metric space** if

1. $d(x,y) = d(y,x)$
2. $d(x,y) = 0$ *iff* $x = y$
3. $d(x,y) \le d(x,z) + d(z,y)$ *(triangle inequality)*

<u>Def 1.2</u> A sequence $\{x_n\}$ **converges** to x, if for every $\epsilon > 0$ there exists N, such that for all $n \ge N$,

$$d(x, x_n) \le \epsilon.$$

<u>Def 1.3</u> A sequence $\{x_n\}$ is called **Cauchy** (or **fundamental**) if for every $\epsilon > 0$ there exists N, such that

$$d(x_m, x_n) \le \epsilon, \qquad \text{for all} \qquad m, n \ge N.$$

<u>Def 1.4</u> A metric space is called **complete** if every Cauchy sequence converges.

<u>Def 1.5</u> A metric space is called **compact** if any sequence has a converging subsequence.

<u>Def 1.6</u> Series

$$\sum_{n=1}^{\infty} x_n \tag{1}$$

converges if its partial sums, $S_N = \sum_{n=1}^{N} x_N$, converge as $N \to \infty$.

<u>Def 1.7</u> Series (1) **converges absolutely** if

$$\lim_{N\to\infty} \sum_{n=1}^{N} |x_n| < \infty.$$

<u>Def 1.8</u> Given a power series, $\sum_{n=0}^{\infty} c_n z^n$, define

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}; \qquad R = \frac{1}{\alpha}.$$

R is called the **radius of convergence** of the series, the latter converges if $|z| < R$ and diverges if $|z| > R$.

1.1. CONVERGENCE TESTS

Root test. Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|x_n|}$. Series (1) converges (absolutely) or diverges if $\alpha < 1$ or $\alpha > 1$ respectively. (More analysis is required if $\alpha = 1$)

Ratio test. Series (1) converges (absolutely) if $\limsup_{n\to\infty} |x_{n+1}/x_n| < 1$ and diverges if there exists some number N such that $|x_{n+1}/x_n| \geq 1$ for all $n > N$.

Comparison test. If $\lim_{n\to\infty} \left|\frac{x_n}{y_n}\right| = C \in (0,\infty)$, series $\sum_{n=1}^{\infty} x_n$ converges absolutely iff series $\sum_{n=1}^{\infty} y_n$ does.

1.2. IS THERE A "BOUNDARY" BETWEEN CONVERGING AND DIVERGING SERIES?

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$$

converges for $\alpha > 1$ and diverges for $\alpha \leq 1$. Thus the exponent $\alpha = 1$ corresponds to the "boundary" for power-law decay rates between converging and diverging series. However, for more general functions, how "close" can we get to $1/n$ while still maintaining convergence? For example,

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^\alpha}$$

converges for all $\alpha > 1$ and diverges for $\alpha \leq 1$. So we lifted our boundary a bit, from $1/n$ to $1/(n \ln n)$. We can go even further and observe that

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n (\ln \ln n)^\alpha}$$

converges for all $\alpha > 1$ and diverges for $\alpha \leq 1$. Etc, etc: we can keep adding more iterated logarithms (or other functions) in a similar manner. Is there some limit to this process? In other words, e.g, is there some special monotone-decreasing sequence $\{b_n\}$ such that whenever $c_n/b_n \to 0$ (as $n \to \infty$) the series $\sum c_n$ converges and whenever $b_n/d_n \to 0$, the series $\sum d_n$ diverges?

1.3. FUN STUFF

Consider the geometric series,

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \qquad |z| < 1.$$

Pretending that this formula is valid for arbitrary $z \neq 1$, we can "derive" that $1 - 1 + 1 - 1 + \cdots = 1/2$, or $1 + 2 + 4 + 8 + \cdots = -1$. In this case the divergent sum acquires meaning via analytic continuation of some appropriately chosen function outside of the radius of convergence of its power series. In a similar fashion one can get such formulas as, e.g.,

$$1 - 2 + 3 - 4 + \cdots := \left.\frac{1}{(1+z)^2}\right|_{z=1} = \frac{1}{4}; \qquad 1 + 2 + 3 + 4 + \cdots := \zeta(-1) = -\frac{1}{12}.$$

2. CONTINUITY AND DIFFERENTIATION

Unless specified otherwise, we consider functions between metric spaces $\mathcal{X}$ and $\mathcal{Y}$.

DEF 2.1 A function f is called **continuous at x_0** if for every $\epsilon > 0$ there exists $\delta > 0$, such that for all $x \in \mathcal{X}$ with $d_{\mathcal{X}}(x, x_0) < \delta$, $d_{\mathcal{Y}}\big(f(x), f(x_0)\big) < \epsilon$. A function which is continuous at every point of $\mathcal{X}$ is called **continuous in $\mathcal{X}$.**

DEF 2.2 A function f is called **uniformly continuous** if for every $\epsilon > 0$ there exists $\delta > 0$, such that for all $x_1, x_2 \in \mathcal{X}$ with $d_{\mathcal{X}}(x_1, x_2) < \delta$, $d_{\mathcal{Y}}\big(f(x_1), f(x_2)\big) < \epsilon$.

Assume that our metric spaces are also *normed* linear vector spaces with metric and norm related via $d(f, g) = \|f - g\|$.

DEF 2.3 Suppose $\mathcal{O}$ is an open set in $\mathcal{X}$; f maps $\mathcal{O}$ into $\mathcal{Y}$; $x_0 \in \mathcal{O}$. If there exists a *bounded* linear operator $\mathbf{D}f(x_0)$, such that

$$\lim_{\|x\|_{\mathcal{X}} \to 0} \frac{\|f(x_0 + x) - f(x_0) - \mathbf{D}f(x_0)x\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}} = 0, \tag{2}$$

then f is called **differentiable at x_0**, and $\mathbf{D}f(x_0)$ is called the **(Fréchet) derivative** or **differential** of f at x_0. If f is differentiable at every point in $\mathcal{O}$, we call f differentiable in $\mathcal{O}$. The *determinant* of the operator $\mathbf{D}f(x_0)$ (if well-defined) is called the **Jacobian** of f at x_0.

2.1. SOME IMPORTANT RESULTS

Mean value theorem. Suppose f is continuous on $[a, b]$ and differentiable in (a, b). There exists $x \in (a, b)$, such that

$$f'(x) = \frac{f(a) - f(b)}{a - b}.$$

Taylor's theorem (1d). Suppose $f \in \mathrm{C}^{n-1}[a, b]$ and $f^{(n)}(x)$ exists for all $x \in (a, b)$. For all x and y such that $a < x < y < b$, there exists $\xi \in [x, y]$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(y)}{k!}(x - y)^k + \frac{f^{(n)}(\xi)}{n!}(x - y)^n.$$

Taylor's theorem (multi-d). Let $\bar{\mathcal{B}}$ be a closed ball centered at the origin in $\mathbb{R}^n$; $f \in \mathrm{C}^n(\bar{\mathcal{B}})$; $x \in \mathcal{B}$. Then

$$f(x) = \sum_{|\alpha| < n} \frac{x^\alpha}{\alpha!}\, \partial^\alpha f(0) + \sum_{|\alpha| = n} \frac{x^\alpha}{\alpha!}\, \partial^\alpha f(\xi x), \qquad \text{for some} \qquad \xi \in [0, 1].$$

Inverse function theorem. Assume that f is a continuously differentiable function from $\mathbb{R}^n$ and $\mathbf{D}f(x)$ is invertible. Then f is invertible in some neighborhood of x and its inverse is continuously differentiable in some neighborhood of $f(x)$.

Implicit function theorem. Assume that F is a continuously differentiable function from (an open subset) $\mathcal{O} \subset \mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^m$; $(x, y) \in \mathcal{O}$; $F(x, y) = 0$; and $\mathbf{D}F$ is one-to-one. Then there exists a neighborhood $\mathcal{N} \subset \mathbb{R}^n$ containing x and a function $f : \mathcal{N} \to \mathbb{R}^m$, such that $f(x) = y$ and $F\big(x, f(x)\big) = 0$ for all $x \in \mathcal{N}$.

3. Integration, theorems relating integrals and derivatives

DEF 3.1 A finite ordered subset of $[a,b]$, $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$, such that

$$a = \pi_1 < \pi_2 < \ldots < \pi_{n-1} < \pi_n = b$$

is called a **partition** of $[a,b]$. We say $\boldsymbol{\pi}_2$ is a **refinement** of $\boldsymbol{\pi}_1$ if $\boldsymbol{\pi}_1 \subset \boldsymbol{\pi}_2$. A sequence of partitions $\{\boldsymbol{\pi}^n\}$ is called **fine** if each partition in the sequence is a refinement of the previous one and

$$\lim_{n\to\infty} \max_{k=2,\ldots,|\boldsymbol{\pi}^n|} (\pi_k^n - \pi_{k-1}^n) = 0.$$

DEF 3.2 Suppose the functions f and g are such that following limit exists and is the same for all fine sequences of partitions of $[a,b]$ and all $\boldsymbol{x}(\boldsymbol{\pi}) = (x_2, \ldots, x_{|\boldsymbol{\pi}|})$ such that $x_k \in [\pi_{k-1}, \pi_k]$, $k = 2, \ldots, |\boldsymbol{\pi}|$:

$$\lim_{n\to\infty} \sum_{k=2}^{|\boldsymbol{\pi}^n|} f\big(x_k(\boldsymbol{\pi}^n)\big)\big|g(\pi_k^n) - g(\pi_{k-1}^n)\big|.$$

It is then called the **Riemann-Stieltjes integral** of f with respect to g over $[a,b] =: \Omega$ and is denoted by

$$\int_a^b f(x)\,\mathrm{d}g(x) \qquad \text{or} \qquad \int_\Omega f\,\mathrm{d}g. \tag{3}$$

- If g is differentiable, then Riemann-Stieltjes integral can be related to the usual Riemann integral,

$$\int_\Omega f\,\mathrm{d}g = \int_\Omega f(x)g'(x)\,\mathrm{d}x.$$

DEF 3.3 A function $f : [a,b] \to \mathbb{R}$ is called of **bounded variation** if

$$\mathsf{V}_a^b(f) := \sup_{\boldsymbol{\pi}\in\mathcal{P}[a,b]} \sum_{n=2}^{|\boldsymbol{\pi}|} |f(\pi_n) - f(\pi_{n-1})| < \infty.$$

Here $\mathcal{P}[a,b]$ denotes the set of all partitions of $[a,b]$. The space of all functions of bounded variation on $[a,b]$ is denoted by $\mathsf{BV}[a,b]$.

3.1. Some Important Results

Existence of Riemann-Stieltjes integral Suppose $f \in \mathsf{C}[a,b]$ and $g \in \mathsf{BV}[a,b]$, then the Riemann-Stieltjes integral (3) exists.

- For a given $g \in \mathsf{BV}[a,b]$, the class of functions integrable with respect to g is larger than $\mathsf{C}[a,b]$ and essentially includes all Riemann-integrable functions which do not share points of discontinuity with g.

Fundamental theorem of calculus. Let $f \in \mathsf{C}[a,b]$, $g \in \mathsf{BV}[a,b]$, then

$$\int_a^b \mathrm{d}g = g(b) - g(a); \quad \text{if in addition } g \in \mathsf{C}[a,b], \quad \text{then} \quad \frac{\mathrm{d}}{\mathrm{d}g}\int_a^x f(y)\,\mathrm{d}g(y) = f(x) \quad \text{for all } x \in [a,b].$$

Here $\dfrac{\mathrm{d}F(x)}{\mathrm{d}g(x)} := \lim_{\epsilon\to 0} \dfrac{F(x+\epsilon) - F(x)}{g(x+\epsilon) - g(x)}$ — (essentially) the **Radon-Nikodym derivative** of F with respect to g.

Change of variables. Suppose $g, h \in \mathsf{BV}(\Omega)$; $f, \mathrm{d}g/\mathrm{d}h \in \mathsf{C}[a,b]$, then

$$\int_\Omega f\,\mathrm{d}g = \int_\Omega f \frac{\mathrm{d}g}{\mathrm{d}h}\,\mathrm{d}h.$$

Integration by parts. Suppose $f, g \in \mathsf{BV}[a,b]$, $f \in \mathsf{C}[a,b]$, then

$$\int_a^b f\,\mathrm{d}g = f(b)g(b) - f(a)g(a) - \int_a^b g\,\mathrm{d}f.$$

Integral mean value theorem I. Let f be continuous and g monotone on $[a,b]$, then there exists $x \in [a,b]$, such that

$$\int_a^b f\,\mathrm{d}g = f(x)\big[g(b) - g(a)\big].$$

Integral mean value theorem II. Let f be monotone and g be continuous on $[a,b]$, then there exists $x \in [a,b]$, such that

$$\int_a^b f\,\mathrm{d}g = f(a)\big[g(x) - g(a)\big] + f(b)\big[g(b) - g(x)\big].$$

4. SEQUENCES OF FUNCTIONS

DEF 4.1 A sequence of functions $\{f_n\}$ converges to f **pointwise** in $\mathcal{X}$ if for every $x \in \mathcal{X}$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

DEF 4.2 A sequence of functions $\{f_n\}$ converges to f **uniformly** in $\mathcal{X}$ if for every $\epsilon > 0$ there exists N such that for all $n > N$ and all $x \in \mathcal{X}$,

$$d\big(f_n(x), f(x)\big) < \epsilon.$$

DEF 4.3 A family of functions, $\mathcal{F}$, is called **equicontinuous** if for all $\epsilon > 0$ there exists $\delta > 0$, such that whenever $d_{\mathcal{X}}(x_1, x_2) < \delta$,

$$d_{\mathcal{Y}}\big(f(x_1), f(x_2)\big) < \epsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

4.1. SOME IMPORTANT RESULTS

Weierstrass M-test. If $\sup_{x\in\mathcal{X}} |f_n(x)| < M_n$ and the series $\sum M_n$ converges, then $\sum f_n(x)$ converges uniformly in $\mathcal{X}$.

Uniform convergence theorem. A uniform limit of continuous functions is continuous.

Monotone convergence theorem. A point-wise monotone sequence of continuous functions converging to a continuous function on a compact set does so uniformly.

Exchanging the order of limits and integration. Suppose f_n converge uniformly to f in Ω and each f_n is integrable with respect to g over Ω, then

$$\lim_{n\to\infty} \int_\Omega f_n\,\mathrm{d}g = \int_\Omega f\,\mathrm{d}g.$$

Exchanging the order of limits and differentiation. Suppose f_n' converge uniformly on $[a,b]$ and f_n converge at some $x_0 \in [a,b]$, then f_n converge uniformly on $[a,b]$ to some differentiable function f and

$$\lim_{n\to\infty} f_n'(x) = f'(x).$$

Stone-Weierstrass theorem. Continuous functions on $\mathbb{R}^n$ may be uniformly approximated by polynomials on compact subsets of $\mathbb{R}^n$.

Arzelà-Ascoli Theorem. Every infinite equicontinuous family of maps between compact metric spaces contains a uniformly converging sequence.