ALGEBRA QUALIFYING EXAMINATION

JANUARY 2020

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A. Find the minimal polynomial of

$$\begin{pmatrix} 0 & & & & a_0 \\ 1 & 0 & & & \vdots \\ & 1 & \ddots & & \vdots \\ & & \ddots & 0 & a_{n-2} \\ & & & 1 & a_{n-1} \end{pmatrix}$$

Here $a_0, \cdots, a_{n-1}$ are complex numbers.

1B. Find the determinant of the following rational n by n matrix for $n \geq 3$.

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{pmatrix}$$

2A. Determine the group of automorphisms of the symmetric group S_3.

2B. Show that the symmetric group S_4 is isomorphic to $\mathrm{GL}_2(\mathbb{F}_3)/Z(\mathrm{GL}_2(\mathbb{F}_3))$, where $\mathbb{F}_3$ is the field with three elements.

3A. Let A be a unital commutative ring and P be a prime ideal of A. If $I_1, \dots, I_n$ are ideals of A and $P = \cap_{i=1}^n I_i$, then $P = I_i$ for some i.

3B. (1) Prove the following statement: If R is a commutative ring, then the set of nilpotent elements of R is an ideal of R.

(2) Prove or disprove: if R is an arbitrary ring then the set of nilpotent elements of R is an ideal of R.

4A. Determine the Galois group of $x^n - t \in \mathbb{C}(t)[x]$ over $\mathbb{C}(t)$.

4B. Let F be a field of characteristic $p > 0$ and $a \in F$. Consider $f(x) = x^p - x - a$. Assume that $f(x)$ is irreducible, determine the Galois group of $f(x)$.

5A. Find (up to isomorphism) all semisimple rings with 324 elements.

5B. Let A be a unital commutative ring and M be a noetherian module over A. Assume the following condition: if $a \in A$ and $am = 0$ for all $m \in M$, then $a = 0$. Show that A is a noetherian ring.