ALGEBRA QUALIFYING EXAMINATION

SPRING 2024

1A. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, let $||x|| = \sqrt{\sum_{i=1}^n x_i^2}$. For a linear map $A\colon \mathbb{R}^n \to \mathbb{R}^n$, set $||A|| = \sup_{||x||=1} ||Ax||$ (the supremum is taken over all $x \in \mathbb{R}^n$ such that $||x|| = 1$). Suppose that $||A|| < 1$. Prove that $A + I$, where I is the identity map, is invertible.

1B. Let

$$M = \begin{pmatrix} 4 & 2 & -2 & 5 & -1 \\ 0 & 4 & 0 & 2 & 3 \\ 0 & 0 & 4 & 2 & 3 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

be a matrix over $\mathbb{C}$. Find the Jordan canonical form of M.

2A. Recall that the exponent of a finite group G is the minimal positive integer k such that $x^k = e$ for all $x \in G$. Suppose a group G has order 12 and exponent 12. Prove that G has a subgroup of index 2.

2B. Let G be a finite group, H a normal subgroup of G, and P a Sylow p-subgroup of G. Prove that $P \cap H$ is a Sylow p-subgroup of H.

3A. Suppose $\mathbb{F}$ is a field, and $p \in \mathbb{F}[x]$ is a degree n polynomial which has n distinct roots in $\mathbb{F}$. Prove that the ring $\mathbb{F}[x]/(p)$ is isomorphic to $\mathbb{F}^n = \mathbb{F} \oplus \cdots \oplus \mathbb{F}$.

3B. Let R be a commutative ring with 1. Prove that if $R[x]$ is a PID, then R is a field.

4A. Consider the polynomial $p(x) = x^4 - 3x^2 + 3$.

(a) Let $\pm\alpha, \pm\beta$ be its roots. Calculate $\alpha^2\beta^2(\alpha^2 - \beta^2)^2$.

(b) Prove that the Galois group of $p(x)$ over $\mathbb{Q}[\sqrt{-1}]$ is cyclic.

4B. Let $K = F(\alpha)$ be a Galois extension of F, with $\alpha \notin F$. Suppose there exists $\sigma \in \mathrm{Gal}(K/F)$ such that $\sigma(\alpha) = \alpha^{-1}$. Prove that the degree of the extension $[K : F]$ is even and $[F(\alpha + \alpha^{-1}) : F] = \frac{1}{2}[K : F]$.

5A. Find invariant factors of the $\mathbb{Z}$-module $M = (\mathbb{Z}^2 \oplus \mathbb{Z}_6) \otimes_{\mathbb{Z}} (\mathbb{Z} \oplus \mathbb{Z}_4)$, i.e. integers $d_1 \mid \cdots \mid d_n$ such that $M \simeq \mathbb{Z}/(d_1) \oplus \cdots \oplus \mathbb{Z}/(d_n)$.

5B. Let R be a commutative ring, and M be a Noetherian R-module. Suppose $f : M \to M$ is a surjective homomorphism. Prove that f is an isomorphism.