ANALYSIS QUALIFYING EXAM

AUGUST 2021

Please show all of your work. GOOD LUCK!

(1) Evaluate
$$\int_0^1 \left(\frac{\ln(1-x)}{x^2} + \frac{1}{x}\right) dx.$$
Justify your steps. *Hint:* You may find using Taylor series useful.

(2) Let f be a real valued continuously differentiable function on the real line. Suppose that f' is bounded, $f(0) = 0$, and $f'(0) = 5$. Find
$$\lim_{n\to\infty} n^2 \int_0^\infty f(x) e^{-n^2x^2} dx.$$
Justify all steps.

(3) Let $f \in L^2([a,b])$ be real-valued. Show that
$$\sqrt{\left(\int_a^b f(x)^2 \cos(x)\, dx\right)^2 + \left(\int_a^b f(x)^2 \sin(x)\, dx\right)^2} \le \int_a^b f(x)^2\, dx\,.$$
Hint: One may, for example, write $f(x)^2 \cos(x) = f(x) \cdot f(x) \cos(x)$.

(4) Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and let $f(x)$ be a measurable function on X. Suppose that
$$m(t) = \mu\{x :\ |f(x)| > t\} = \frac{1}{t^4}$$
for $t > 1$. Find all values of p, $1 \le p \le \infty$ such that $f(x) \in L^p(X, \mu)$.

(5) Let $\{f_n\}_{n=1}^\infty$ be an orthonormal basis of $L^2([0,1])$. For each $p > 0$ and any $t \in [0,1]$, calculate
$$\sum_{n=1}^\infty \left| \int_0^t s^p f_n(s)\, ds \right|^2 .$$

(6) Let $f(x) = (\sin x)/x$. Define a measure μ on $\mathbb{R}$:
$$\mu(X) = m(\{x :\ f(x) \in X\})$$
for every Borel set X. Here m is the Lesbegue measure. Prove that μ is absolutely continuous with respect to m and find $(d\mu/dm)(3/\pi)$. Here $d\mu/dm$ is the Radon–Nikodym derivative. *Hint.* $f(\pi/6) = 3/\pi$.