ANALYSIS QUALIFYING EXAM

JANUARY 2020

Please show all of your work.

(1) For any $n \geq 1$, let $f_n : [0,1] \to \mathbb{R}$ be a continuous function.

i) Define what it means to say that f_n converges uniformly to f.

ii) Prove that if f_n converges uniformly to f, then f is continuous.

iii) Show by example that if f_n converges to f pointwise, it is not necessarily the case that f is continuous.

(2) Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and $\{E_n\}_{n=1}^{\infty}$ be a sequence of measurable sets, i.e. $E_n \in \mathcal{M}$ for all $n \geq 1$. Prove that

$$\mu\left(\liminf E_n\right) \leq \liminf \mu(E_n)$$

Note: for a sequence of subsets $E_n \subset X$,

$$\liminf E_n = \bigcup_{N=1}^{\infty} \left(\bigcap_{n=N}^{\infty} E_n \right)$$

(3) Let $(\mathcal{X}, \mathcal{M}, \mu)$ be a measure space and f a measurable function on $\mathcal{X}$. Suppose that

$$\sum_{n=0}^{\infty} \int_{\mathcal{X}} |f(x)|^n \, d\mu(x) < \infty$$

i) Show that $|f(x)| < 1$ μ-almost everywhere.

ii) Prove that

$$g(x) = \frac{1}{1 - f(x)}$$

is integrable with respect to μ.

(4) Prove that the operator $A : L^2([0,1]) \to L^2([0,1])$ given by

$$(Af)(x) = \int_0^1 \frac{f(t)}{\sqrt{|x-t|}} \, dt$$

is bounded. **Hint:** You may want to use that $\sqrt{|x-t|} = (|x-t|)^{1/4}(|x-t|)^{1/4}$.

(5) For each $n \geq 1$, let f_n be Lebesgue measurable on $[0, 1]$.

i) Define what it means to say that f_n converges to zero in measure.

ii) Prove that if each f_n is integrable and $\int_0^1 |f_n(x)| dx \to 0$ as $n \to \infty$, then f_n converges to zero in measure.

iii) Show by example that $\int_0^1 |f_n(x)| dx \to 0$ as $n \to \infty$ does not necessarily imply that f_n converges to zero pointwise almost everywhere.

(6) Prove or disprove the following claim: The subset of $C([0, 1])$ consisting of absolutely continuous functions ϕ with $\phi(0) = 0$ and

$$\int_0^1 |\phi'(x)|^p \, dx \leq 1$$

is precompact, i.e. has compact closure, in $C([0, 1])$, provided $p > 1$.