ANALYSIS QUALIFYING EXAM

JANUARY 2022

Please show all of your work. GOOD LUCK!

(1) Let $f_n(x)$ be a sequence of differentiable functions on the interval $[0,1]$. Suppose that $f_n(x)$ converge to a function $f(x)$ uniformly on $[0,1]$ and that for every $x \in [0,1]$ the limit of $f_n'(x)$ exists. Prove or give the counter example to the following statement: the function $f(x)$ is differentiable.

(2) Let $f(x)$ be a bounded, continuously differentiable function on $[0,\infty)$. Suppose that $f(0)=0$ and $f'(0)=2$. Evaluate

$$\lim_{n\to\infty} n^2 \int_0^\infty f(x)e^{-nx}dx.$$

Justify all steps.

(3) Let $f(x)$ be an absolutely continuous function on the interval $[0,1]$, and let $f(0)=0$. Prove that

$$\int_0^1 \frac{|f(x)|^2}{x}dx \leq \int_0^1 (1-x)|f'(x)|^2dx.$$

(4) Let $(X,\mathcal{M},\mu)$ be a measure space. Suppose that $\mu(X)=1$. Let $f(x)$ be a real-valued measurable function on X, and let

$$\alpha(t) = \mu\{x : \ -t < f(x) < t\}.$$

Prove that

$$\int_X |f(x)|^2 d\mu = 2\int_0^\infty t(1-\alpha(t))dt.$$

(5) Find all positive values of α for which the function

$$f(x) = x\cos\left(x^{-\alpha}\right)$$

is of bounded variation on the interval $[0,1]$.

(6) Let $(X,\mathcal{M},\mu)$ be a measure space. Suppose that $\mu(X)=1$.Let $\phi_1(x),\ldots,\phi_n(x)$ be an ortho-normal system of real valued functions in $L^2(X,\mu)$. Let

$$f(t) = \int_X \cos\left[t\sum_{j=1}^n \phi_j(x)\right]d\mu(x).$$

Find

$$\lim_{t\to 0} \frac{1 - f(t)}{t^2}.$$

Justify all steps