ALGEBRA QUALIFYING EXAMINATION

AUGUST 2025

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A Let A be an invertible (complex) 2×2 matrix. Show that there is a (complex) matrix B such that $A = B^2$.

1B Consider the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

(a) Is A diagonalizable over $\mathbb{C}$?

(b) Is A diagonalizable over $\mathbb{F}_2$?

2A List all group homomorphisms from S_3 to $\mathbb{C}^\times$. You need to provide a brief justification of your list.

2B Count abelian groups of order 72 which admit a surjective homomorphism to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

3A Let R be a commutative unital ring, and assume that $R[x]$ is noetherian. Is R necessarily noetherian? Justify your answer.

3B Let $f : R \to S$ be a surjective ring homomorphism.

(a) If R is a field, is S necessarily a field?

(b) If R is an integral domain, is S necessarily an integral domain?

4A Show that the field extension E/F is Galois, and determine its Galois group. Here

$$E = \mathbb{C}(x, y, z), \quad F = \mathbb{C}(x + y + z, xy + yz + zx, xyz).$$

4B Let E/F be a Galois extension of Galois group A_4. Show that E admits no Galois subextension of degree 4 over F.

5A Give an example of a non-semisimple module over $\mathbb{F}_2[G]$ where $G = \mathbb{Z}/2\mathbb{Z}$.

5B Prove $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.