ANALYSIS QUALIFYING EXAM
AUGUST, 2025

Please show all your work

GOOD LUCK!

Problem 1

Find

$$\lim_{n\to\infty} n\int_0^\infty \ln(1+x)e^{-nx^2}dx.$$

Justify all steps.

Problem 2

Let $u_n(x)$ be a sequence of continuous functions on the real line that converges to a function $u(x)$ uniformly. Prove that the sequence of functions $\arctan(u_n(x))$ converges to $\arctan(u(x))$ uniformly.

Problem 3

Let $u(x)$ be a solution to the differential equation

$$u'' = (\sin x)u^2$$

on the interval $[0,1]$, such that $u'(0) = 0$. Prove that

$$\int_0^1 |u'(x)|^2 dx \le \frac{1}{12}\int_0^1 |u(x)|^4 dx.$$

Problem 4

Let $(X, \mathfrak{M}, \mu)$ be a finite measure space, and let $f(x)$ be a measurable real-valued function on X. Let

$$\alpha(t) = \mu(\{x : f(x) > t\}).$$

Prove that

$$\int_X \arctan(f(x))d\mu = \int_{-\infty}^\infty \frac{\alpha(t)}{1+t^2}dt - \frac{\pi}{2}\mu(X).$$

Hint. Consider the set $\{(x,t) \subset X\times\mathbb{R} : f(x) > t\}$.

Problem 5

Prove that the function the function $f(x) = x^\alpha \sin x$ is not of bounded variation on the interval $(0,\infty)$ for all real values of α.

Problem 6

Let $e_k = (0, \dots, 0, 1, 0, \dots)$, with 1 sitting in the kth place. Let

$$x_n = \frac{e_1 + \cdots + e_n}{\sqrt{n}}.$$

a) Prove that the seqence x_n converges weakly to 0 in l^2.
b) Does it converge weakly in l^1? Prove your answer.