GEOMETRY-TOPOLOGY QUALIFYING EXAM, F2025

1. Suppose that $f(z)$ is a 1-1 complex analytic function in an open neighborhood of $z = 0$ with $f(0) = 0$. Show that $\frac{f(z)}{z}$ has a removable singularity at $z = 0$ and is non-zero near $z = 0$.

2. Show that there does NOT exist a 1-form ω on $\mathbb{R}^n$ such that for any (smooth) curve $c : [0, 1] \to \mathbb{R}^n$ $\int_c \omega$ is the arclength of c.

3. Let (r, θ) denote polar coordinates for the plane $\mathbb{R}^2$, where the coordinate neighborhood is $\mathbb{R}^2$ minus the nonnegative x-axis.

(a) Compute the form $d\theta$ in terms of the Euclidean coordinates (x, y), and use this to explain why $d\theta$ extends to a smooth one form on $\mathbb{R} \setminus \{0\}$.

(b) Compute the integrals $\int_C d\theta$ for the oriented curves C in figure 1 (see attached sheet).

(c) Explain why $d\theta$ is not exact in all of $\mathbb{R}^2 \setminus \{0\}$.

4. Suppose that M is a smooth n-manifold. Prove that for $1 \le p \le n$,

$$H^p_{DR}(M, \mathbb{R}) \times H_p(M, \mathbb{R}) \to \mathbb{R} : ([\omega], [c]) \to \int_c \omega$$

is a well-defined bilinear map. Here, $[\omega]$ is a class of closed forms in the de Rham cohomology space $H^p_{DR}(M, \mathbb{R})$, and $[c]$ is a class of cycles in the homology space $H_p(M, \mathbb{R})$.

5. (a) Compute the fundamental group (at a basepoint which you can choose) of the triangle with three sides identified as in figure 2.

(b) Explain why the fundamental group, up to isomorphism, does not depend on your choice of basepoint.

6. For each of the following statements, either briefly explain why the statement is true, or give a counterexample.

(a) Every exact k-form on a compact orientable k-dimensional manifold vanishes at some point.

(b) Suppose that $f : X \to Y$ is a smooth mapping of manifolds. If f is $1-1$ and onto, then f is a diffeomorphism.

(c) If the degree of a smooth map $f : S^2 \to S^2$ is nonzero, then the map f is onto.

Figure 1

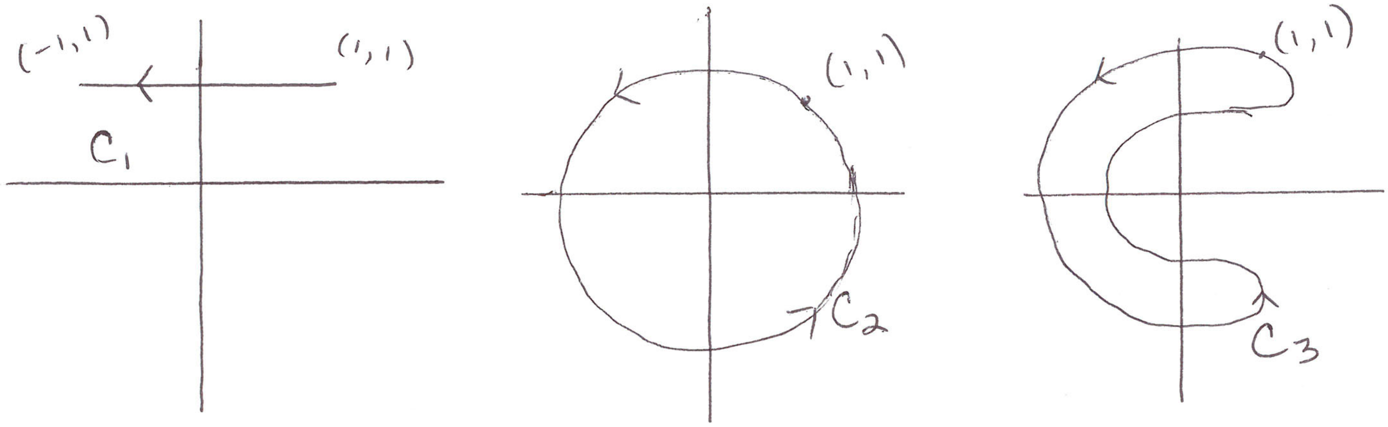

Figure 2