ALGEBRA QUALIFYING EXAMINATION

JANUARY 2026

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A Let A be a square complex matrix, such that ${}^tA = \overline{A}$. Show that all eigenvalues of A are real.

1B Given a prime number p, find the number of 2×2 matrices A over $\mathbb{F}_p$ such that A^2 is the zero matrix.

2A Let p be a prime, and G be a p-group, i.e. $|G|$ is a power of p. Show that the number of nonnormal subgroups of G is divisible by p.

2B Prove that, up to isomorphism, there exists a unique nonabelian group of order 18 with a cyclic subgroup of order 9.

3A Let A be a commutative noetherian ring with 1. (1) If I is an ideal, then A/I is also an noetherian ring. (2) Given an example of a noetherian ring A and subring B which is not noetherian.

3B Show that every maximal ideal of $\mathbb{C}[x, y]$ is not principal.

4A By considering the extension $\mathbb{Q}(e^{2\pi i/7})/\mathbb{Q}$, explain that $(\cos \frac{2\pi}{7})^2$ is not a rational number.

4B Let K be the splitting field of $x^4 - 5x^2 + 6$ over $\mathbb{Q}$.
   (a) Determine the Galois group of K over $\mathbb{Q}$.
   (b) Determine all subextensions of K over $\mathbb{Q}$.

5A (1) Explain using the structure theorem of finitely generate modules over PIDs that if a finitely generated abelian group is torsion free, then it is free. (2) Given an example of a module over $\mathbb{Z}[x]$ that is torsion free but not free. You need to justify your example.

5B Let M and N be nonzero modules over an integral domain R.
   (a) Is $\mathrm{Hom}_R(M, N)$ necessarily a nonzero R-module?
   (b) Is $M \otimes_R N$ necessarily a nonzero R-module?