# ANALYSIS QUALIFYING EXAM
## JANUARY, 2026

Please show all your work

GOOD LUCK!

### Problem 1

Find

$$\lim_{n\to\infty} \sqrt{n} \int_0^1 e^{-nx^2} \cos x dx.$$

Justify all steps.

### Problem 2

Let $u_n(x)$ be a sequence of uniformly continuous functions on the real line that converges to a function $u(x)$ uniformly. Prove that the function $u(x)$ is uniformly continuous.

### Problem 3

Let $f(x)$ be an absolutely continuous, real valued function on the interval $[0, 1]$. Suppose that

$$\int_0^1 f(x)dx = 0.$$

Prove that

$$\int_0^1 |f(x)|^3 dx \leq \frac{1}{3} \int_0^1 |f'(x)|^3 dx.$$

### Problem 4

Let $(X, \mathfrak{M}, \mu)$ be a finite measure space, and let $f(x)$ be a measurable real-valued function on $X$. Prove that the function

$$g(x,t) = \frac{\sin(tf(x))}{1+t^2}$$

is integrable on $X \times \mathbb{R}$, with the product measure $\mu \times m$, where $m$ is the Lesbegue measure on $\mathbb{R}$. Evaluate the integral

$$\int_{X\times\mathbb{R}} g(x,t)d(\mu \times m).$$

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## Problem 5

Let $f(x)$ be an absolutely continuous function on $\mathbb{R}$. Prove that the function $\sin(f(x))$ is also absolutely continuous.

## Problem 6

Let $A$ be a bounded operaror in a Hilbert space $H$, and let $x \in H$. Suppose that $||A|| < 1$ and $(A^n x, x) = 0$ for every positive, integer number $n$. Prove that

$$((I - A)^{-1}x, x) = ||x||^2.$$