GEOMETRY AND TOPOLOGY QUALIFYING EXAM, JANUARY 2026

1. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function on the open unit disc.

(a) Show that the function $g(z) = \frac{f(z)-f(-z)}{z}$ has a removable singularity at $z = 0$.

(b) Using (a) show that

$$\sup_{z\in\mathbb{D}} |f(z) - f(-z)| \geq 2|f'(0)|$$

2. Let X be a compact, connected topological space, $T : X \to X$ a continuous mapping, and $f : X \to \mathbb{R}$ a continuous function. Prove that there exists $x \in X$ such that $f(T(x)) = f(x)$. *Hint: Consider the set of values of the function* $f(T(x)) - f(x)$.

3. Let $x^2 + y^2 = 1$ and $z^2 + w^2 = 1$ be the unit circles in (two copies of) $\mathbb{R}^2$, with standard orientations. Consider their Cartesian product $\{(x, y, z, w) : x^2 + y^2 = 1, z^2 + w^2 = 1\}$ as a submanifold T^2 of $\mathbb{R}^4$ with the standard product orientation. Compute

$$\int_{T^2} \omega$$

where $\omega = xyz\, dw \wedge dy$.

4. Define $\Phi : \mathbb{R}^4 \to \mathbb{R}^2$ by

$$\Phi(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$

Show that $(0, 1)$ is a regular value of Φ, and that the level set $\Phi^{-1}((0, 1))$ is diffeomorphic to the two-dimensional sphere S^2.

5. The Klein bottle can be defined as a square $[0, 1] \times [0, 1]$ with $(s, 0)$ identified with $(s, 1)$ and $(0, t)$ identified with $(1, 1 - t)$ for every $0 \leq s, t \leq 1$. In terms of this realization, find a presentation for the fundamental group (at some base point).

6. (a) Using calculus only, show that a one-form η on S^1 is exact if and only if

$$\int_{S^1} \eta = 0.$$

(b) Using Mayer-Vietoris sequence applied to de Rham cohomology, show that a two-form ω on S^2 is exact if and only if

$$\int_{S^2} \omega = 0.$$