

GRADUATE TOPICS COURSE PROPOSAL FOR SPRING 2027:
Introduction to modular forms and Galois representations

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Course Description

This is a frankencourse introducing students both to modular forms and to Galois representations, with emphasis on the deep connections between these two subjects. Galois representations are a natural way to understand symmetries of extensions of the rational numbers through the imprint these symmetries leave on matrix groups, especially after class field theory clarified the one-dimensional case around a century ago. Modular forms are complex-analytic functions with many symmetries of their own; their study goes back to Ramanujan, also around a hundred years ago. The surprise is that these analytic functions encode deep arithmetic information: a modular form that is an eigenform for Hecke operators has an attached two-dimensional Galois representation so that the Hecke eigenvalues of the form correspond to traces of the action of Frobenius elements in the representation. Moreover, every two-dimensional Galois representation satisfying some reasonable properties is expected to arise in this way.

Either of these topics could merit a course in their own right, of course. But our graduate students have limited time and need to learn many things quickly — perhaps especially now in the confusing nascent age of AI. In addition to beginning the study of both topics, this course serves as an introduction to some of the ideas underlying the Langlands program. The central theme is that apparently different kinds of mathematical objects can encode the same arithmetic information. This course will set up students to begin developing this dictionary in the concrete setting of modular forms and two-dimensional Galois representations, where the theory is both deep enough to reveal some of the central ideas of modern number theory and concrete enough to be accessed in a single semester.

Prerequisites

Students must be comfortable with algebra through Galois theory and modules (MA 415a/b or possibly 511a/b or equivalent) as well as with basic topology (MA 432 or equivalent). A background in basic algebraic number theory (especially number fields, prime splitting and ramification, Frobenius elements, as in MA 514a or equivalent) is extremely desirable, but an ambitious student may be able to pick up enough to make do. Representation theory of finite groups (often covered in 511b and/or 517a) and familiarity with the statements of class field theory (MA 514a/b or equivalent) would be very helpful, but will be reviewed.

Approximate schedule

Part I: Introduction to modular forms (6 weeks)

- Action of $SL_2(\mathbb{Z})$ on the upper half-plane, congruence subgroups, cusps
- Modular forms, cusp forms
- Fourier expansions and the q -expansion principle
- Dimension formulas

- Level 1: Eisenstein series, Δ , structure of the ring of modular forms
- Hecke operators; eigenforms
- Newforms and oldforms, Atkin-Lehner theory
- Hecke eigenfields and eigenorders
- L -functions associated to eigenforms

Part II: Introduction to Galois representations (6 weeks)

- Infinite Galois groups and Krull topology
- Absolute Galois groups of finite fields, local fields (review of tame ramification), number fields with limited ramification
- Review of representation theory for finite groups over complex numbers
- Brauer-Nesbitt theorem and when the trace function suffices
- Review of Chebotarev density theorem.
- Galois representation definitions
- Artin representations
- p -adic cyclotomic character
- Tate module of an elliptic curve
- Statement of Deligne's theorem on Galois representations attached to modular forms.
- Hecke characters
- Local representations, briefest hint of p -adic Hodge theory
- Residual Galois representations (stable lattices, semisimplification)
- Modular mod- p representations and Serre's Conjecture / Serre Reciprocity
- Modularity of two-dimensional Galois representations (Fontaine-Mazur conjecture)

Additional topics and/or possibly student projects (3 weeks)

- Pseudocharacters and pseudorepresentations
- The eigenconstruction and Hecke algebras
- Deformation theory
- Modularity of elliptic curves (the $R = \mathbb{T}$ idea)
- Serre-Swinnerton-Dyer theory of using residual Galois representations to explain congruences of coefficients of modular forms
- Ribet's converse to Herbrand's theorem

Textbooks and sources

The course will draw selectively from several references, including but not limited to...

- Serre, *Linear Representations of Finite Groups*
- Milne, *Algebraic Number Theory*
- Milne, *Class Field Theory*
- Neukirch, *Algebraic Number Theory*.

- Diamond and Shurman, *A First Course in Modular Forms*.
- Serre, *A Course in Arithmetic*.
- Milne, *Modular Functions and Modular Forms*.
- Koblitz, *Introduction to Elliptic Curves and Modular Forms*
- Zagier, “Elliptic Modular Forms and Their Applications”, in *The 1-2-3 of Modular Forms*.
- Silverman, *The Arithmetic of Elliptic Curves*
- Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*.
- Wiese, “Galois representations”.
- Taylor, “Galois representations”.
- Mazur, “How can we construct abelian Galois extensions of basic number fields?”
- Swinnerton-Dyer, “On ℓ -adic representations and congruences for coefficients of modular forms”

Learning outcomes

Upon completion of the course, students should be able to...

1. Define modular forms for congruence subgroups and work with their Fourier expansions.
2. Describe the action of Hecke operators and explain the significance of Hecke eigenforms.
3. Begin to describe absolute Galois groups of finite, local, and number fields.
4. Analyze decomposition groups, inertia groups, and Frobenius elements of Galois groups.
5. Understand why complex Galois representation have finite image.
6. Understand the construction of the p -adic cyclotomic character and how to work with it.
7. Understand the construction of the Tate module of an elliptic curve as a Galois representation and begin to understand its properties.
8. State Deligne’s theorem on the existence of Galois representations associated to modular eigenforms.
9. Understand the construction of L -functions associated with modular forms, elliptic curves, and Galois representations.