ALGEBRA QUALIFYING EXAMINATION

AUGUST 2026

Do either one of nA or nB for $1 \leq n \leq 5$. Justify all your answers.

1A Compute $A^{10} - A^4 + A^2$ for the matrix

$$A = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}.$$

1B Let V be a nonzero finite dimensional real vector space and $J : V \to V$ be a linear operator such that $J^2 = -1$ (here 1 means the identity map).
   (a) Prove that V has a two dimensional J-invariant subspace.
   (b) Show that $\dim V$ is even.

2A Given three distinct prime numbers p, q, r, prove that every group of order pqr is not simple.

2B Let G be a group and C be its center.
   (a) Show that if G/C is cyclic, then G must be abelian.
   (b) Give a nonabelian group G such that G/C is abelian.

3A Consider the ring $R = \mathbb{Q}[x, y, z]/(x^2 - yz)$.
   (a) Determine whether R is a unique factorization domain.
   (b) Show that the ideal generated by x, y is a prime ideal in R.

3B Show that

$$\mathbb{C}[x, y]/(y^2 - x^2 + 1)$$

is an integral domain and its field of fractions is isomorphic to $\mathbb{C}(t)$.

4A Let f be an irreducible polynomial over $\mathbb{Q}$ of odd degree. If the Galois group of f is cyclic, prove that all roots of f are real numbers.

4B Let $\zeta = e^{2\pi i/13}$. Determine the Galois group of $\mathbb{Q}(\zeta + \zeta^3 + \zeta^9)/\mathbb{Q}$.

5A Let $A = \mathbb{C}[t^2, t^3]$ and $B = \mathbb{C}[t]$ viewed as an A-module. View $\mathbb{C}$ as an A-module on which t^2 and t^3 act by 0. Determine the structure of

$$\mathrm{Hom}_A(B, \mathbb{C})$$

as a B-module, i.e. write it as a direct sum of cyclic B-modules.

5B Let $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x^2]$ viewed as a subring of A. Let

$$M = \mathbb{C}[x]/(x^2 + x), \quad N = \mathbb{C}[x]/(x^2 - 1).$$

Compute the dimensions of $M \otimes_A N$ and $M \otimes_B N$ as vector spaces over $\mathbb{C}$.