Analysis qualifier — August 2026

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Directions: Read carefully and provide complete, well written, well explained and organized answers.

Explicitly state any basic results or major theorems used, and verify their applicability.

1. Let $(X, d(\cdot,\cdot))$ be a metric space, $M \subset X$ be compact and $A \subset M$ be dense in M. Prove that for every $\delta > 0$ there exists a finite subset $A_\delta \subset A$ such that for every $x \in M$ there exists at least one $a \in A_\delta$ with $d(x, a) \leq \delta$.

2. Consider a measurable space $(\Omega, \mathcal{F})$ and a finitely additive measure μ on $\mathcal{F}$. Show that μ is countably additive on $\mathcal{F}$ if and only if for all increasing sequences of sets $\{A_n\}_{n=1}^{\infty}$ (that is $A_n \subset A_{n+1}\ \forall\, n$) in $\mathcal{F}$ and $A := \cup_n A_n$, it holds that $\lim_{n\to\infty} \mu(A_n) = \mu(A)$.

3. Determine the following limit. Justify your reasoning

$$\lim_{n\to\infty} \int_0^\infty \left(\frac{1+x^2}{x^2+x+1}\right) ne^{-nx}\, dx$$

4. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions of bounded variation, converging pointwise to f for all $x \in [a, b]$, and $0 < M < \infty$ be a constant such that

$$\mathrm{Var}\,(f_n, [a, b]) \leq M \qquad \text{for all } n\,.$$

   Prove that f is of bounded variation and that $\mathrm{Var}\,(f, [a, b]) \leq M$.

5. Let E be a measurable set and $f : E \to \mathbb{R}$ be a measurable function with respect to the measure μ, and assume that $0 < r < s < \infty$ are given. Prove that

$$\left(\int_E |f|^r\, d\mu\right)^{1/r} \leq \left(\int_E |f|^s\, d\mu\right)^{1/s} \mu\,(E)^{1/r-1/s}\,.$$

6. Let H be a separable Hilbert space, $\{e_n\}_{n=1}^{\infty} \subset H$ be an orthonormal basis, and $\{\delta_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be a sequence of positive real numbers. Define

$$Q := \left\{ x \in H \,\middle|\, x = \sum^{\infty} c_n e_n \qquad \text{with } |c_n| \leq \delta_n \right\}.$$

   Prove that Q is compact if and only if $\displaystyle\sum_{n=1}^{\infty} \delta_n^2 < \infty$.