GEOMETRY-TOPOLOGY QUALIFYING EXAM, AUGUST 2026

1. Find a conformal mapping from the upper half of the unit disk, $\{z \in \mathbb{C} : |z| < 1, Im(z) > 0\}$, to the upper half plane, $H^2 = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$.

2. (a) Determine the critical points and critical values for the function

$$f : \mathbb{R}^3 \to \mathbb{R} : \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto x^3 + y^2 - yz - y$$

(b) Let $M = f^{-1}(0)$. Find an explicit smooth parameterization for an open neighborhood of $\vec{0} \in M$.

(c) Find a basis for the tangent space of M at the point $\vec{0}$, viewed as a subspace of $\mathbb{R}^3$.

3. Consider the two form

$$\omega := (x^2 + y^2 + z^2)^{-3/2}(xdy \wedge dz - ydx \wedge dz + zdx \wedge dy) \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$$

Calculate the integral of ω over the boundary of the cube $\{(x, y, z) \in [-1, 1] \times [-1, 1] \times [-1, 1]\}$.

4. Let X denote the space obtained by deleting two lines for $\mathbb{R}^3$,

$$X = \mathbb{R}^3 \setminus (l_1 \cup l_2)$$

where l_1 is the y-axis and l_2 is the x-axis translated by e_3, $l_2 := \{(x, 0, 1) : x \in \mathbb{R}\}$. Calculate the fundamental group of X (at a basepoint of your choice).

5. Let $H^2 = \{\tau \in \mathbb{C} : Im(\tau) > 0\}$, the upper half of the complex plane. Consider the map

$$P : H^2 \to \mathbb{C} : \tau \mapsto q = exp(2\pi i\tau)$$

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(a) Find the image of the map P.

The map P defines a covering space for the image (you do not need to show this).

(b) Find the group of automorphisms of the covering space.

(c) Find a maximal open elementary neighborhood in the base, and a corresponding fundamental domain for the covering map (An open elementary neighborhood U in the base has the property that P restricts to a homeomorphism of each path connected component of $P^{-1}(U)$. A fundamental domain is a subset $D \subset H^2$ such that each point H^2 is equivalent by an automorphism of the covering to a unique

point in D).

6. Compute the integral homology, and the DeRham cohomology, of the surface $(S^1 \times S^1) \setminus \{q\}$ (where q is a point on the torus). Briefly justify each calculation (by stating any general theorem which you use), and in particular indicate explicit generators for each class (for convenience you can view the torus as the quotient $\mathbb{R}^2/\mathbb{Z}^2$, and pick the point q in any way you wish).