Congruence properties modulo prime powers for a class of partition functions

Abstract

Let p be prime, and let $p_{[1,p]}(n)$ denote the function whose generating function is $\prod(1-q^n)^{-1}(1-q^{pn})^{-1}$. This function and its generalizations $p_{[c^\ell,d^m]}(n)$ are the subject of study in several recent papers. Let $\ell \geq 5$, let $j \geq 1$, and let $p \in \{2,3,5\}$. In this paper, we prove that the generating function for $p_{[1,p]}(n)$ in the progression $\beta_{p,\ell,j}$ modulo ℓ^j with $24\beta_{p,\ell,j} \equiv p+1 \pmod{\ell^j}$ lies in a Hecke-invariant subspace of type $\{\eta(Dz)\eta(Dpz)F(Dz) : F(z) \in M_s(\Gamma_0(p),\chi)\}$ for suitable $D \geq 1$, $s \geq 0$, and character χ. When $p \in \{2,3,5\}$, we use the Hecke-invariance of these subspaces to prove that for distinct primes ℓ and $m \geq 5$ and $j \geq 1$, congruences of the form

$$p_{[1,p]}\left(\frac{\ell^j m^k n + 1}{D}\right) \equiv 0 \pmod{\ell^j}$$

for all $n \geq 1$ with $m \nmid n$, where k is explicitly computable and depends on the forms in the invariant subspace. This is a joint work with Matthew Boylan.