An Unstable Instance of the Finite Element Method, continued

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Introduction

Recall that the Finite Element Method is seeking a solution to a partial differential equation. The process is two steps:

1. Define a space of solutions based on a mesh
2. Select the “best” answer

FEM Errors

There are three types of errors in this process:

1. **Approximation:** Is the space of solutions dense enough?
2. **Consistency:** Do the equations used in selection work consistently?
3. **Stability:** Is the model well-posed? Does there exist a **unique** solution?

Note: There does not seem to be a clear consensus on the definition of the term “stability”.

Outline

Goals for this talk:

1. Explain a specific instance of a “non-stable” finite element method system.
2. Explain how stability can be achieved by the use of polynomial differential forms

The source for both parts is the report “Finite element exterior calculus, homological techniques, and applications” by Arnold, Falk, and Winther; Acta Numerica 2006

Overview

For a particular finite-element method problem, we will define spaces of functions X and X^1 such that:

$$X^1 \subsetneq X \quad \text{that is} \quad X^1 \quad \text{is a closed}^* \text{ proper subset of} \quad X$$

We will show that the finite-element-selected solution must lie in X^1 while the exact solution lies in X.

Thus **any approximation by the finite element method cannot converge to the exact solution.**

*In this context, "closed" means with respect to limits of sequences of functions.

Poisson problem

The weak formulation of the vector Poisson problem is to minimize the energy of the integral:

$$\int \frac{|\text{div}|^2}{2} + \frac{|\text{curl}|^2}{2} - f(x)u(x)dx$$

over vector fields satisfying some boundary condition, e.g. $u \cdot \eta = 0$, where f and η are given functions.

Standard FEM: Minimize energy over piecewise polynomial functions of degree at most K, a subspace of all possible solutions.

Poisson problem

Translated into the language of differential forms, we seek to minimize the quadratic functional:

$$J(u) = \frac{1}{2} < du, du > + \frac{1}{2} < \delta u, \delta u > - < f, u >$$

over functions u for which J is defined.

$$\left(\text{Compare: } \int \frac{|\text{div}|^2}{2} + \frac{|\text{curl}|^2}{2} - f(x)u(x)dx \right)$$

Now we will clarify the definitions involved.

Differential Forms

- A **differential k-form on a manifold** M is a function ω that assigns to each point $x \in M$ an alternating k-tensor $\omega(x)$ on the tangent space of M at x

$$\omega : M \rightarrow \Lambda^k \left(T_x(M)^*\right)$$

  (Recall: M denotes a manifold, $\Lambda^k(V)$ is the set of alternating k-tensors on the vector space V, and $T_x(M)$ is the tangent space of M at the point x.)
- Thus, for a point $x \in M$, $\omega(x)$ is itself a function:

$$\omega(x) : (\mathbb{R}^n)^K \rightarrow \mathbb{R}$$

  such that $\omega(x)$ is multilinear and alternating. (Here we assume M is an n-manifold embedded in some $\mathbb{R}^N$)

Geometrical meaning of forms

Example from Hubbard and Hubbard: Vector Calculus, Linear Algebra, and Differential Forms

Consider the 2-form $dx_1 \wedge dx_2$ and the vectors $\vec{a}, \vec{b} \in \mathbb{R}^3$. Then

$$dx_1 \wedge dx_2 \left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right) = \det \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

So this form gives the (x_1, x_2) component of signed area. So

$$dx_i(\vec{v}) = v_i = \text{the } i\text{th component of the signed length of } \vec{v}$$

A general form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ evaluated on k vectors gives the $(x_{i_1}, \ldots, x_{i_k})$ component of signed k-dimensional volume of the k-parallelogram spanned by the vectors.

Geometrical meaning of forms

Notably, on an *n*-manifold, there is a unique *n*-dimensional form (up to multiplication by functions).

$$\mathbf{vol} = dx_1 \wedge \cdots \wedge dx_n$$

It is called the **volume form** and unsurprisingly calculates the (*n*-dimensional) volume of the region over which it is integrated.

Exterior Derivative

The **exterior derivative** d is an operator that **increases** form degree:

$$d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$$

Example: Let $\omega = xy^2 dx$. Then

$$d\omega = \left(\frac{\partial}{\partial x} xy^2\right) dx \wedge dx + \left(\frac{\partial}{\partial y} xy^2\right) dx \wedge dy = 0 + 2xydxdy$$

Formally, we have

$$d\omega_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} \partial_{v_j} \omega_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1})$$

Coderivative

The **coderivative** δ is an operator that **decreases** form degree:

$$\delta : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$$

Short definition for operator theorists: δ is the formal adjoint of d with respect to the L^2-inner product. That is, if ω or η vanishes near the boundary of Ω then $< d\omega, \eta > = < \omega, \delta\eta >$.

What does that mean? I'm glad you asked.

The Hodge Star Operator

The **Hodge Star** $\star$ is an operator that turns k tensors to $n - k$ tensors:

$$\star : \mathrm{Alt}^k V \to \mathrm{Alt}^{n-k} V$$

where V is a vector space of dimension n. How does it work? Let $\omega \in \mathrm{Alt}^k V$. Then there exists a linear map L_ω:

$$\begin{array}{ll} L_\omega : \mathrm{Alt}^{n-k} \to \mathbb{R} & L_\omega = \pi \circ \phi \\ \phi : \mathrm{Alt}^{n-k}\Omega \to \mathrm{Alt}^n\Omega & \mu \mapsto \omega \wedge \mu \\ \pi : \mathrm{Alt}^n\Omega \to \mathbb{R} & c\mathbf{vol} \mapsto c \end{array}$$

where **vol** denotes the unique basis n-tensor. Then there exists $\star\omega \in \mathrm{Alt}^{n-k}$ such that

$$L_\omega(\mu) = < \star\omega, \mu >$$

The Hodge Star Operator

The inner product $< \cdot, \cdot >$ of two k-tensors returns an element of $\mathbb{R}$ via

$$< \alpha, \beta > = \sum_{\sigma} \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \beta(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

where the sum is taken over increasing sequences $\sigma : \{1, \ldots, k\} \rightarrow \{1, \ldots, n\}$ and $v_1, \ldots, v_n$ is an orthonormal basis.

So, in words, $\star\omega$ is the $n - k$ tensor such that the inner product of it and any other $n - k$ tensor μ times the **vol** tensor equals $\omega \wedge \mu$:

$$< \star\omega, \mu > \textbf{vol} = \omega \wedge \mu \quad \mu \in \text{Alt}^{n-k} V$$

This definition extends in a natural way to a star operator on forms.

Another Inner Product

Let Ω be an n-manifold. Then given two k forms ω,ν on Ω, we have the L^2-**inner product** given by

$$< \omega, \nu >_{L^2\Lambda^k} = \int_\Omega < \omega_x, \nu_x > \textbf{vol} = \int \omega \wedge \star\nu$$

This allows us to create the Hilbert space $L^2\Lambda^k\Omega$.

The spaces X and X^1

Now we're ready to define the spaces for our example. Let Ω be a domain in $\mathbb{R}^n$ and $0 \leq k \leq n$ an integer. Then we define

$$L^2\Lambda^k\Omega = \text{completion of } \Lambda^k\Omega \text{ w.r.t. inner product norm}$$

We also define the spaces:

$$H\Lambda^k\Omega = \left\{\omega \in L^2\Lambda^k\Omega : d\omega \in L^2\Lambda^{k+1}\Omega\right\}$$

$$H^*\Lambda^k\Omega = \left\{\omega \in L^2\Lambda^k\Omega : \delta\omega \in L^2\Lambda^{k-1}\Omega\right\}$$

$$\mathring{H}^*\Lambda^k\Omega = \left\{\omega \in H^*\Lambda^k\Omega : \omega \text{ has compact support}\right\}$$

The spaces X and X^1

We define the space X to be

$$X = H\Lambda^k\Omega \cap \mathring{H}^*\Lambda^k\Omega$$

In words, X is the set of k-forms ω in L^2 such that $d\omega$ and $\delta\omega$ are also in L^2 and $\delta\omega$ has compact support.

The spaces X and X^1

$$X = H\Lambda^k\Omega \cap \mathring{H}^*\Lambda^k\Omega$$

Note that our FEM-selected functions are piecewise smooth and therefore belong to the subspace

$$X^1 = H^1\Lambda^k\Omega \cap \mathring{H}^*\Lambda^k\Omega$$

where

$$H^1\Lambda^k\Omega = \left\{\omega \in L^2\Lambda^k\Omega : \text{first partials of } \omega \text{ are all in } L^2\right\}$$

The spaces X and X^1

$$X = H\Lambda^k\Omega \cap \mathring{H}^*\Lambda^k\Omega$$

$$X^1 = H^1\Lambda^k\Omega \cap \mathring{H}^*\Lambda^k\Omega$$

We claim that X^1 is closed, proper subset of X. That is

- Any convergent sequence of forms in X^1 will have a limit in X^1
- There exists $\omega \in X - X^1$ i.e. a form whose exterior derivative is square integrable but whose partial derivatives are not. (Such a function must involve some cancelation of the “bad” partials).

To make matters worse, the authors claim that “except for very non-generic data, the solution u to the Hodge Laplacian problem will belong to X but **not** to X^1.”

A stable solution

It can be shown that the spaces of polynomial differential forms can be used to create a stable solution, under certain weak hypotheses. The authors prove this by

- Defining the spaces of polynomial differential forms
- Characterizing the Hodge-Laplacian problem in a “mixed formulation”
- Citing a source* that provides an equivalent condition for stability, then proving that condition.

*Source: I Babuska and A.K. Aziz, Survey lectures on the mathematical foundations of the finite element method; in The Mathematical Foundations of the FEM with Applications to PDEs (Proc. Sympos., Univ. Maryland; Academic Press, New York, 1972) pp. 1-359

Polynomial Differential Forms

Recall the following definitions:

$$P_r\mathbb{R}^n = \{\text{polynomials in } n\text{-variables of deg} \leq r \text{ with } \mathbb{R} \text{ coefficients}\}$$

$$P_r\Lambda^k\mathbb{R}^n = \{f\omega\} \quad \text{where } \omega \text{ is a } k\text{-form on } \mathbb{R}^n \text{ and } f \in P_r\mathbb{R}^n$$

This is a discrete space with dimension

$$\left|P_r\Lambda^k\mathbb{R}^n\right| = \left(\begin{array}{c} n+r \\ n \end{array}\right)\left(\begin{array}{c} n \\ k \end{array}\right) = |P_r\mathbb{R}^n|\left|\Lambda^k\mathbb{R}^n\right|$$

Polynomial Differential Forms

Now we define a finite element space for a simplicial complex T_h

$P_r\Lambda^k(T_h) =$ piecewise polynomial differential forms which restrict to $P_r\Lambda^k(T)$ on each n-simplex $T \in T_h$.

Why would we expect these spaces make a finite element method stable? Perhaps because differential forms can be integrated "without recourse to any additional structure, such as a measure or metric, on a manifold Ω." (AFW)

Loosely, dealing with forms keeps us in the theoretical realm for longer, thereby delaying the accumulation of errors.

A Stability Criterion

The Babuska-Brezzi “inf-sup” stability criterion says that a **unique** solution **exists** to certain PDEs if the following holds:

There exists a constant L such that

$$\inf_p \sup_v \frac{(p, \mathrm{div} v)}{|Dv||p|} > L > 0$$

where $p \in L^2$, $v \in H^1$ (i.e. $v' \in L^2$). Here

$$(p, \mathrm{div} v) = \int p \,\mathrm{div} v, \quad |Dv| \text{ is the } H^1 \text{ semi-norm}, \quad |p| = \left(\int |p|^2 \right)^{1/2}$$

The authors demonstrate the stability of the FEM for the following problem:

Find $\sigma_h \in \Lambda_h^{k-1}, u_h \in \Lambda_h^k, p_j \in \mathfrak{H}_h^k$ such that

$$\begin{aligned} < \sigma_h, \tau > &= < d\tau, u_h >, & \tau \in \Lambda_h^{k-1} \\ < d\sigma_h, v > + < du_h, dv > + < v, p_h > &= < f, v >, & v \in \Lambda_h^k \\ < u, q > &= 0, & q \in \mathfrak{H}_h^k \end{aligned}$$

where $\mathfrak{H}^k = \left\{ \omega \in H\Lambda^k \cap \mathring{H}^* \Lambda^k : d\omega = 0, \delta\omega = 0 \right\}$, the space of harmonic forms. Note that the subscript h denotes we are considering forms for some fixed triangulation T_h.

The Criterion Applied

Stability can be proven for the discretized space

$$P^{r+1}\Lambda^{k-1}(T_h) \times P_r\Lambda^k(T_h)$$

and other spaces of affine-invariant polynomial differential forms, which we omit for now.

The Criterion Applied

The stability of the method for this space is equivalent to the inf-sup condition:

There exist constants $\gamma > 0$, $C < \infty$ independent of h such that for any $(\sigma, u, p) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$ there exists $(\tau, v, q) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$ with

$$B(\sigma, u, p; \tau, v, q) \geq \gamma(||\sigma||_{H\Lambda}^2 + ||u||_{H\Lambda}^2 + ||p||^2),$$

$$||\tau||_{H\Lambda} + ||v||_{H\Lambda} + ||q|| \leq C(||\sigma||_{H\Lambda} + ||u||_{H\Lambda} + ||p||).$$

where B is the bounded bilinear form

$$\begin{aligned} B(\sigma, u, p; \tau, v, q) &= <\sigma, \tau> - <d\tau, u> + <d\sigma, v> + \\ &\quad <du, dv> + <v, p> - <u, q> \end{aligned}$$

References

- Arnold, Falk, and Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numerica 2006
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- Hubbard and Hubbard, Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach