The Geometry of Interpolation Error Estimates for Isogeometric Analysis

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joint work with Chandrajit Bajaj and Alexander Rand (UT Austin)

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Suppose we want to solve a finite element problem with

**geometry map** \( T : \) reference element → physical elements

**interpolation map** \( I : \) physical DoFs → interpolated function

If \( T \) is non-affine, *a priori* error estimates for \( I \) can converge at sub-optimal rates.

**ARNOLD, BOFFI, FALK** *Approximation by Quadrilateral Finite Elements*, Math. Comp., 2002

\[ ||u - I_\ell u||_{H^1(\Omega)} \leq c h ||u||_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega) \]

\[ ||u - I_q u||_{H^1(\Omega)} \leq c h^2 ||u||_{H^3(\Omega)}, \quad \forall u \in H^3(\Omega) \]

\[ ||u - I_s u||_{H^1(\Omega)} \leq c h ||u||_{H^3(\Omega)}, \quad \forall u \in H^3(\Omega) \]

How can the usual \( O(h^2) \) rate for the serendipity element be recovered for non-affine \( T \)?

**Isogeometric solution:** Define \( I \) on physical elements; characterize geometric dependence.
Approach

Use **Generalized Barycentric Coordinates (GBCs)** to create polygonal finite elements.

**Advantages of the approach**

- Builds on rich theory of **GBCs** from graphics literature, amenable to IGA.
- Some **GBC** elements are not sensitive to large angles, allowing remeshing:

  ![Diagram](image1)

- Provides canonical adaptive meshing elements:

  ![Diagram](image2)
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Definition

Let $\Omega$ be a convex polygon in $\mathbb{R}^2$ with vertices $v_1, \ldots, v_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on $\Omega$ if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on $\Omega$.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(v_i) \lambda_i$.

Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity**: $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision**: $\sum_{i=1}^{n} v_i \lambda_i(x) = x$.
5. **Interpolation**: $\lambda_i(v_j) = \delta_{ij}$.

**Theorem [Warren, 2003]**

If the $\lambda_i$ are rational functions of degree $n - 2$, then they are unique.
Many generalizations to choose from . . .

- Triangulation
  \[ 0 \leq \lambda_i^{T_m}(x) \leq \lambda_i(x) \leq \lambda_i^{T_M}(x) \leq 1 \]

- Wachspress
  \[ \Rightarrow WACHSPRESS, \ A \ Rational \ Finite \ Element \ Basis, \ 1975. \]

- Sibson / Laplace
  \[ \Rightarrow SIBSON, \ A \ vector \ identity \ for \ the \ Dirichlet \ tessellation, \ 1980. \]
  \[ \Rightarrow HIYOSHI, \ SUGIHARA, \ Voronoi-based \ interpolation \ with \ higher \ continuity, \ 2000. \]
Many generalizations to choose from . . .

- **Mean value**

- **Harmonic**

Many more in graphics contexts...
Comparison via ‘eye-ball’ norm

Triangulated

Wachspress

Mean Value
Comparison via ‘eyeball’ norm

Wachspress

Sibson

Mean Value

Discrete Harmonic
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Let $\Omega$ be a convex polygon with vertices $v_1, \ldots, v_n$.

For linear elements, an **optimal convergence estimate** has the form

$$
\left\| u - \sum_{i=1}^{n} u(v_i)\lambda_i \right\|_{H^1(\Omega)} \leq C \operatorname{diam}(\Omega) \|u\|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega).
$$

(1)

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in $H^1$ norm.


For (1), it suffices to prove an **$H^1$-interpolant estimate** over domains of diameter one:

$$
\left\| \sum_{i=1}^{n} u(v_i)\lambda_i \right\|_{H^1(\Omega)} \leq C_I \|u\|_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega).
$$

(2)

For (2), it suffices to **bound the gradients** of the $\{\lambda_i\}$, i.e. prove $\exists C_\lambda \in \mathbb{R}$ such that

$$
\|\nabla \lambda_i\|_{L^2(\Omega)} \leq C_\lambda.
$$

(3)
To bound the gradients of the coordinates, we need control of the element geometry.

Let $\rho(\Omega)$ denote the radius of the largest inscribed circle. The **aspect ratio** $\gamma$ is defined by

$$
\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)
$$

Three possible geometric conditions on a polygonal mesh:

- **G1. BOUNDED ASPECT RATIO:** $\exists \gamma^* < \infty$ such that $\gamma < \gamma^*$
- **G2. MINIMUM EDGE LENGTH:** $\exists d_* > 0$ such that $|\mathbf{v}_i - \mathbf{v}_{i-1}| > d_*$
- **G3. MAXIMUM INTERIOR ANGLE:** $\exists \beta^* < \pi$ such that $\beta_i < \beta^*$
Ex: Relation of Mean Value Interpolation to Geometry

\[ \lambda_i^{MV}(x) := \frac{w_i(x)}{\sum_{j=1}^{n} w_j(x)} \]

\[ w_i(x) := \frac{\tan\left(\frac{\alpha_i(x)}{2}\right) + \tan\left(\frac{\alpha_{i-1}(x)}{2}\right)}{||v_i - x||} \]

Main Problem: \( \nabla w_i \) is complicated. Just part of the expression is

\[ \nabla \left[ \frac{\tan\left(\frac{\alpha_i(x)}{2}\right)}{r_i(x)} \right] = \frac{\sec^2\left(\frac{\alpha_i(x)}{2}\right) \nabla \alpha_i / 2}{r_i(x)} - \frac{\tan\left(\frac{\alpha_i(x)}{2}\right)}{(r_i(x))^2} \nabla r_i(x) \]

Solution: Divide analysis into six cases based on proximity to \( v_a \) and size of \( \alpha_i \) and \( \alpha_j \)

Rand, Gillette, Bajaj Interpolation Error Estimates for Mean Value Coordinates, submitted
Theorem

In the table, any necessary geometric criteria to achieve the $H^1$ interpolant estimate are denoted by N. The set of geometric criteria denoted by S in each row taken together are sufficient to guarantee the $H^1$ interpolant estimate.

<table>
<thead>
<tr>
<th></th>
<th>$G_1$ (aspect ratio)</th>
<th>$G_2$ (min edge length)</th>
<th>$G_3$ (max interior angle)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangulated</td>
<td>$\lambda^{\text{Tri}}$</td>
<td>-</td>
<td>S,N</td>
</tr>
<tr>
<td>Wachspress</td>
<td>$\lambda^{\text{Wach}}$</td>
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<td>S,N</td>
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<td>Sibson</td>
<td>$\lambda^{\text{Sibs}}$</td>
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<td>-</td>
</tr>
<tr>
<td>Mean Value</td>
<td>$\lambda^{\text{MV}}$</td>
<td>S</td>
<td>-</td>
</tr>
<tr>
<td>Harmonic</td>
<td>$\lambda^{\text{Har}}$</td>
<td>S</td>
<td>-</td>
</tr>
</tbody>
</table>

*Gillette, Rand, Bajaj* Error Estimates for Generalized Barycentric Interpolation
Advances in Computational Mathematics, to appear, 2011

*Rand, Gillette, Bajaj* Interpolation Error Estimates for Mean Value Coordinates, submitted
Outline

1. Background on GBCs
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From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:

\[
\{ \lambda_i \} \xrightarrow{\text{pairwise products}} \{ \lambda_a \lambda_b \}
\]

Why is this naïve?

- For an \( n \)-gon, this construction gives \( n + \binom{n}{2} \) basis functions \( \lambda_a \lambda_b \)
- The space of quadratic polynomials is only dimension 6: \( \{ 1, x, y, xy, x^2, y^2 \} \)
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge \( \Rightarrow \text{only } 2n \text{ functions needed!} \)

Problem Statement

Construct \( 2n \) basis functions associated to the vertices and edge midpoints of an arbitrary \( n \)-gon such that a quadratic convergence estimate is obtained.
We define matrices $A$ and $B$ to reduce the naïve quadratic basis.

**filled dot** = Lagrangian domain point

= all functions in the set evaluate to 0

except the associated function which evaluates to 1

**open dot** = non-Lagrangian domain point

= partition of unity satisfied, but not Lagrange property

\[
\{\lambda_i\} \xrightarrow{\text{pairwise products}} \{\lambda_a \lambda_b\} \xrightarrow{A} \{\xi_{ij}\} \xrightarrow{B} \{\psi_{ij}\}
\]

Linear Quadratic Serendipity Lagrange
The bases are ordered as follows:

- \( \xi_{ii} \) and \( \lambda_a\lambda_a \) = basis functions associated with vertices
- \( \xi_{i(i+1)} \) and \( \lambda_a\lambda_{a+1} \) = basis functions associated with edge midpoints
- \( \lambda_a\lambda_b \) = basis functions associated with interior diagonals, i.e. \( b \notin \{a - 1, a, a + 1\} \)

Serendipity basis functions \( \xi_{ij} \) are a linear combination of pairwise product functions \( \lambda_a\lambda_b \):

\[
\begin{bmatrix}
\xi_{ii} \\
\vdots \\
\xi_{i(i+1)}
\end{bmatrix} = A \begin{bmatrix}
\lambda_a\lambda_a \\
\vdots \\
\lambda_a\lambda_{a+1}
\end{bmatrix} = \begin{bmatrix}
c_{11}^{ij} & \cdots & c_{ab}^{ij} & \cdots & c_{(n-2)n}^{ij} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
c_{11}^{n(n+1)} & \cdots & c_{ab}^{n(n+1)} & \cdots & c_{(n-2)n}^{n(n+1)}
\end{bmatrix} \begin{bmatrix}
\lambda_a\lambda_a \\
\vdots \\
\lambda_a\lambda_{a+1}
\end{bmatrix}
\]
We require the serendipity basis to have quadratic approximation power:

**Constant precision:** \[ 1 = \sum_i \xi_{ii} + 2\xi_{i(i+1)} \]

**Linear precision:** \[ x = \sum_i v_i \xi_{ii} + 2v_{i(i+1)}\xi_{i(i+1)} \]

**Quadratic precision:** \[ xx^T = \sum_i v_i v_i^T \xi_{ii} + (v_i v_{i+1}^T + v_{i+1} v_i^T)\xi_{i(i+1)} \]

**Theorem**

Constants \( \{c_{ij}^{ab}\} \) exist for any convex polygon such that the resulting basis \( \{\xi_{ij}\} \) satisfies constant, linear, and quadratic precision requirements.
Pairwise products vs. Lagrange basis

Even in 1D, pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:

\[ \lambda_0 \lambda_0 \quad \lambda_1 \lambda_1 \]

\[ \lambda_0 \lambda_0 \quad \lambda_0 \lambda_1 \quad \lambda_1 \lambda_1 \]

Translation between these two bases is straightforward and generalizes to the higher dimensional case...
From serendipity to Lagrange

\[
\{ \xi_{ij} \} \xrightarrow{B} \{ \psi_{ij} \}
\]

\[
[\psi_{ij}] = \begin{bmatrix}
\psi_{11} \\
\psi_{22} \\
\vdots \\
\psi_{nn} \\
\psi_{12} \\
\vdots \\
\psi_{n1}
\end{bmatrix} = \begin{bmatrix}
1 & 1 & \cdots & -1 \\
-1 & -1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 1 \\
4 & 4 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
\xi_{11} \\
\xi_{22} \\
\vdots \\
\xi_{nn} \\
\xi_{12} \\
\vdots \\
\xi_{n1}
\end{bmatrix} = B[\xi_{ij}].
\]
Serendipity Theorem

Given bounds on polygon aspect ratio (G1), minimum edge length (G2), and maximum interior angles (G3):

- \(|\|A\||\) is uniformly bounded,
- \(|\|B\||\) is uniformly bounded, and
- \(\text{span}\{\psi_{ij}\} \supset P_2(\mathbb{R}^2) = \text{quadratic polynomials in } x \text{ and } y\)

**Rand, Gillette, Bajaj** *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted*, 2011

* Revised version at my website; will be posted to the arXiV soon.
Using quadratic serendipity GBC interpolation with mean value coordinates:

\[ u_h = l_q u := \sum_{i=1}^{n} u(v_i)\psi_{ii} + u\left(\frac{v_i + v_{i+1}}{2}\right)\psi_{i(i+1)} \]

Non-affine bilinear mapping

<table>
<thead>
<tr>
<th>n</th>
<th>error</th>
<th>rate</th>
<th>error</th>
<th>rate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>5.0e-2</td>
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<td>6.2e-1</td>
<td></td>
</tr>
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<td>6.7e-3</td>
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<td>1.8e-1</td>
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<tr>
<td>64</td>
<td>7.4e-6</td>
<td>2.1</td>
<td>4.96e-3</td>
<td>1.1</td>
</tr>
</tbody>
</table>

Quadratic serendipity GBC method

<table>
<thead>
<tr>
<th>n</th>
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<tbody>
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<td>2.34e-3</td>
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<td>2.22e-2</td>
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<tr>
<td>4</td>
<td>3.03e-4</td>
<td>2.95</td>
<td>6.10e-3</td>
<td>1.87</td>
</tr>
<tr>
<td>8</td>
<td>3.87e-5</td>
<td>2.97</td>
<td>1.59e-3</td>
<td>1.94</td>
</tr>
<tr>
<td>16</td>
<td>4.88e-6</td>
<td>2.99</td>
<td>4.04e-4</td>
<td>1.97</td>
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<td>2.00</td>
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<tr>
<td>256</td>
<td>1.20e-9</td>
<td>3.00</td>
<td>1.64e-6</td>
<td>1.96</td>
</tr>
</tbody>
</table>

\text{Arnold, Boffi, Falk, Math. Comp., 2002}
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From scalar to vector elements

Barycentric functions are used to define $H(\text{curl})$ vector elements on triangles:

\[ \{\lambda_i\} \xrightarrow{\text{Whitney construction}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\} \]

Generalized barycentric functions provide $H(\text{curl})$ elements on polygons:

\[ \{\lambda_i\} \xrightarrow{\text{Whitney construction}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\} \]

This idea fits naturally into the framework of Discrete Exterior Calculus and suggests a wide range of applications... ...work in progress

Gillette, Bajaj  *Dual Formulations of Mixed Finite Element Methods with Applications*  
Basis functions for serendipity spaces

Recent work characterized serendipity spaces in $n$ dimensions for scalar fields, vector fields, and their generalization to differential $k$-forms:

Arnold, Awanou  *Finite Element Differential Forms on Cubical Meshes*  

The results of that paper coupled with the techniques used here suggest a variety of new research directions...  

...work in progress
Recall from the table of geometric dependencies for gradient bounds:

<table>
<thead>
<tr>
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<th>G3 (max interior angle)</th>
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<td>$\lambda^{MV}$</td>
<td>S</td>
<td>S</td>
</tr>
</tbody>
</table>

Thus, the quadratic serendipity construction with mean value coordinates can still allow quadratic convergence for elements with interior angles of $\pi$, in particular, for those used in adaptive methods:

This could help with isogeometric analysis of adaptive methods or T-splines.
References

- Kai Hormann’s webpage on generalized barycentric coordinates:
  http://www.inf.usi.ch/hormann/barycentric

- NSF workshop on generalized barycentric coordinates in NYC (July 25-27):
  http://www.inf.usi.ch/hormann/nsfworkshop

- Slides and pre-prints are available at:
  http://ccom.ucsd.edu/~agillette