Topology-controlled Stable Function Modeling for Biology

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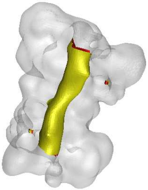
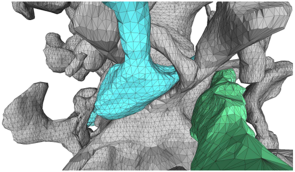
Types of Constraints in Biological Models

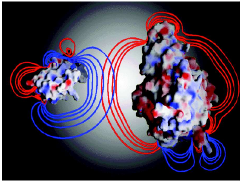
Structural Constraints

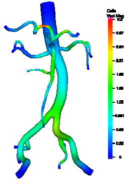




Topological (prescribed genus) | Topological (num. components) | Geometrical (non−intersecting)

Functional Constraints





Satisfy appropriate PDEs

Observations:

- Two surfaces which are geometrically close may have different topological properties (e.g. number of components, presence or absence of a tunnel, etc).
- Functions defined relative to a surface exhibit sensativity to minor changes in topology.

Goals:

- Provably stable strucutral recovery methods satisfying topolgoical and geometrical constraints.
- Functional recovery methods whose error is bounded in terms of error in the domain's structure.

Outline

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1. Prior Work on Functional Constraints
2. Prior Work on Stability
3. A Unifying Framework: Geometric PDEs

Functional Constraints in Biological Modeling

General Functional Modeling Problem

Input: a bounded subset Φ of $\mathbb{R}^3$; a surface $\Omega \subset \Phi$; intensity data $f : \Phi \to \mathbb{R}$ defined relative to Ω.
Output: a scalar function $u : \Phi \to \mathbb{R}$ or vector field $v : \Phi \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ solving a PDE on Φ with input data f.

Ex: Given a surface $\Omega \subset \mathbb{R}^3$ and charge density data q construct

- $u(x)$ = electric potential function at x *(potential energy per unit of charge associated with a static eletric field)*
- $E(x)$ = electric vector field at x *(force per unit charge exerted on a charge placed at x)*
- $B(x)$ = magnetic vector field at x

subject to the constraints of Maxwell's equations:

$$E = -\nabla u, \qquad \nabla \times E = -\frac{\partial B}{\partial t}, \qquad \nabla \cdot B = 0$$

The Poisson Equation

To compute the electric potential u, we must solve

$$-\text{div grad } u \; (= \Delta u = \nabla^2 u) = q$$

where $q : \mathbb{R}^3 \to \mathbb{R}^3$ is a given function representing the charge density, scaled by a permitivity factor depending on the medium.

A *mixed method* approach reformulates this problem as

$$\text{find } \sigma \text{ and } u \text{ such that } \begin{cases} -\text{div } \sigma & = & q \\ \sigma & = & \text{grad } u \end{cases}$$

To approximate u or σ, we must first consider where to look. That is, we must identify the minimum regularity requirements that each variable must satisfy.

Hilbert Spaces, Banach Spaces, and Norms

A *norm* on a space X is a mapping $||\cdot|| : X \to \mathbb{R}$ such that

- $||x|| = 0$ if and only if x is the zero element of X.
- $||rx|| = |r| \cdot ||x||$ for $r \in \mathbb{R}$, $x \in X$.
- $||x + y|| \leq ||x|| + ||y||$.

A *Banach space* X is a complete normed linear space, i.e. X is closed under addition and scalar multiplication, X has a norm, and any sequence of elements of X which is Cauchy in that norm converges to an element of X. An *inner product* on a space X is a mapping $(\cdot,\cdot) : X \times X \to \mathbb{R}$ such that

- $(x, y) = (y, x)$.
- $(x_1 + x_2, y) = (x_1, y) + (x_2, y)$.
- $(rx, y) = r(x, y)$, $\forall r \in \mathbb{R}$.
- $(x, x) \geq 0$ and $(x, x) = 0$ if and only if x is the zero element of X.

An inner product always induces a norm via $||x|| := (x, x)^{1/2}$. A *Hilbert space* is a Banach space which is complete with respect to a norm induced by an inner product on the space. An example of a norm not induced by an inner product is the supremum norm.

Spaces of Functions

There are many Banach spaces of functions that we might consider as our search space.

$$
\begin{aligned}
L^2(\Omega) &= \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \left(\int_\Omega |f^2| \right)^{1/2} < \infty \right\} \\
H^m(\Omega) &= \left\{ \phi \in L^2(\Omega) \;\middle|\; \partial^\alpha \phi \in L^2(\Omega), \quad |\alpha| \le m \right\} \\
&\qquad \text{where } \alpha = (\alpha_1, \alpha_2, \alpha_3) \text{ is a multi-index; } |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \\
H(\mathrm{curl}) &= \left\{ u \in (L^2(\Omega))^3 \;\middle|\; \mathrm{curl}\, u \in (L^2(\Omega))^3 \right\} \\
H(\mathrm{div}) &= \left\{ u \in (L^2(\Omega))^3 \;\middle|\; \mathrm{div}\, u \in L^2(\Omega) \right\}
\end{aligned}
$$

$H^m(\Omega)$ is a simplified version of the **Sobolev Space** or order m. As defined here, $H^m(\Omega)$ is a Banach space with norm

$$
||\phi||_{H^m} := \sum_{|\alpha| \le m} ||\partial^\alpha \phi||_{L^2}
$$

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Written in variational form, integrating by parts, and using the Neumann boundary condition $\partial u / \partial \eta = 0$ yields:

$$\int_\Omega \sigma \cdot \tau \, dx = \int_\Omega \text{div } \tau u \, dx \quad , \quad \forall \tau \in H(\text{div }, \Omega; \mathbb{R}^3)$$

$$\int_\Omega \text{div } \sigma v \, dx = \int_\Omega f v \, dx \quad , \quad \forall v \in L^2(\Omega)$$

Nedelec Spaces

Nedelec's idea (1980)

Mixed finite element methods naturally require families of finite elements conforming in $H(\text{curl})$ and $H(\text{div})$.

H_l = homogeneous polynomials of degree l P_l = polynomials of degree $\leq l$
Let $\vec{v}(x)$ denote a vector field with n component functions $v_1(x), \ldots, v_n(x)$. Then the Nedelec space of the first kind of order $k \geq 1$ is

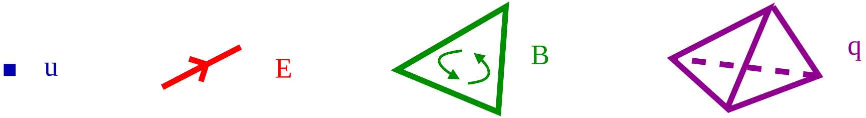
$$R_k = \{\vec{v} : v_i \in P_{k-1}\} \oplus \{\vec{v} : v_i \in H_k \text{ and } \vec{x} \cdot \vec{v}(\vec{x}) = 0, \quad \forall x \in \mathbb{R}^n\}$$

Ex: For $n = 2$, $k = 1$:

$$\begin{aligned}
R_1 &= \{\vec{v} : v_1, v_2 \in P_0\} \oplus \{\vec{v} : v_1, v_2 \in H_1 \text{ and } \vec{x} \cdot \vec{v}(\vec{x}) = 0, \quad \forall \vec{x} \in \mathbb{R}^2\} \\
&= \{< c_1, c_2 >\} \oplus \{< a_1 x + b_1 y, a_2 x + b_2 y >: a_1 x^2 + b_1 xy + a_2 xy + b_2 y^2 = 0\} \\
&= \{< c_1, c_2 >\} \oplus \{< a_1 x + b_1 y, a_2 x + b_2 y >: a_1 = b_2 = 0 \quad \text{and} \quad a_2 = -b_1\} \\
&= \{< c_1 + b_1 y, c_2 - b_1 x >\} \\
&= \left\{\vec{v} : \frac{\partial v_i}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 0, i \neq j, i, j = 1, 2\right\}
\end{aligned}$$

Ref: J.C. Nedelec, ***Mixed Finite Elements in*** $\mathbb{R}^3$, *Numer. Math.* **35**, 315-341, 1980.

Observe that each electromagnetic property previously mentioned is most naturally characterized in a different dimension.



electric potential is point−valued

electric fields are valued based on a linear current flow

magnetic fields are are dual to electric fields and valued on planes

charge density is valued over a volume

We can integrate each variable over a simplex of the associated dimension, suggesting differential forms as the appropriate mathematical description of the phenomena. Formally, let $\Omega \subset \mathbb{R}^3$ be a contractible 3-manifold with smooth boundary. Let $T_x(\Omega)$ denote the tangent space to Ω at x. The associated forms are:

$$\begin{array}{llllll} {}^0u(x): & \emptyset & \longrightarrow \mathbb{R} & \text{via} & \emptyset \mapsto u(x) \\ {}^1E(x): & T_x(\Omega) & \longrightarrow \mathbb{R} & \text{via} & \xi \mapsto E(x)\cdot\xi \\ {}^2B(x): & (T_x(\Omega))^2 & \longrightarrow \mathbb{R} & \text{via} & (\xi,\eta) \mapsto B(x)\cdot(\xi\times\eta) \\ {}^3q(x): & (T_x(\Omega))^3 & \longrightarrow \mathbb{R} & \text{via} & (\xi,\eta,\zeta) \mapsto q(x)(\xi\cdot(\eta\times\zeta)) \end{array}$$

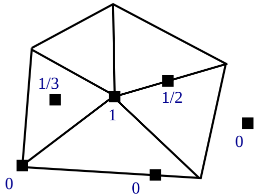
Whitney Forms

Whitney's idea (1957)

Given a simplicial complex, define spaces of k-dimensional forms in terms of their integrals on k-dimensional simplicies.

Given a vertex i in the mesh, let x be a point in one of the tetrahedra that shares vertex i. Define

$$\lambda_i(x) = \text{barycentric weight of the point } x \text{ in its tetrahedron with respect to vertex } i$$

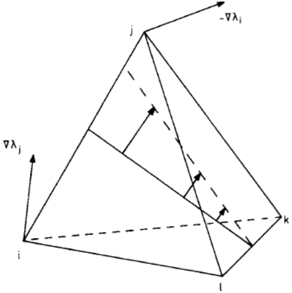


values of λ_i at specified points

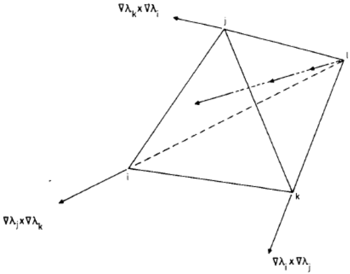
- Barycentric coordinates are defined locally, relative to a tetrahedron (in 3D) or a triangle (in 2D).
- The coordinate systems agree on intersecting simplicies, making λ_i continuous.
- By defining λ_i to be zero on all simplicies not touching vertex i, we turn λ_i into a global function with compact support.

Ref: H. Whitney, ***Geometric Integration Theory***, Princeton University Press, 1957.

Whitney Forms



Whitney edge element w $_{ij}$



Whitney face element w $_{ijk}$

Location	Degree	Notation	Description
vertex *i*	0-form	${}^0w_i$	${}^0\lambda_i$
edge *ij*	1-form	${}^1w_{ij}$	${}^1(\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)$
face *ijk*	2-form	${}^2w_{ijk}$	$2\ {}^2[(\lambda_i\nabla\lambda_j \times \nabla\lambda_k) + (\lambda_j\nabla\lambda_k \times \nabla\lambda_i) + (\lambda_k\nabla\lambda_i \times \nabla\lambda_j)]$
tet *ijkl*	3-form	${}^3w_{ijkl}$	${}^3\chi_{\text{tet}}$ (value 1 inside the tet, 0 elsewhere)

Ref: H. Whitney, ***Geometric Integration Theory***, Princeton University Press, 1957.

Bossavit: Uniting Whitney and Nedelec

Bossavit's idea (1988)

Whitney forms are "geometrical objects that are to differential forms what finite-element interpolating functions are to scalar fields."

In particular, Whitney form spaces have the same essential properties as finite-element spaces used in mixed methods, such as Nedelec's spaces. For example, the exterior derivative operator d works as expected:

$$d\ {}^0u = {}^1(\text{grad}\ u) \quad d\ {}^1E = {}^2(\text{curl}\ E) \quad d\ {}^2B = {}^3(\text{div}\ B) \quad d\ {}^3q = 0$$

and unexpectedly (or rather, by construction) satisfies:

$$d\ {}^pW(\Omega) \subset {}^{(p+1)}W(\Omega)$$

where ${}^pW(\Omega)$ denotes the Whitney forms of order p on Ω. This condition is crucial in our simultaneous search for σ and u satistfying grad $u = \sigma$ and $-$div $\sigma = q$.

Ref: A. Bossavit, ***Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism***, *Proc. of the IEEE*, **135 : 8**, 493-500, 1988

Arnold, Falk and Winther's idea (2006)

Much of the previous work can be phrased in the language of deRham cohomology and this viewpoint helps to characterize stable finite element bases.

Differential forms define the deRham complex:

$$\mathbb{R} \longrightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(\Omega) \longrightarrow 0$$

The kth cohomology group of Ω is defined as:

$$H^k(\Omega) = \ker d / \mathrm{im}\, d$$

The cohomology groups give information about the topology of Ω, e.g.

$$|H^0(\Omega)| = \text{number of components of } \Omega \qquad |H^1(\Omega)| = \text{number of tunnels of } \Omega$$

For stable mixed methods, the key criteria turns out to be that the finite-element spaces "recreate" the deRham complex on the discrete level:

$$\mathbb{R} \longrightarrow \Lambda_h^0(\Omega) \xrightarrow{d_h} \Lambda_h^1(\Omega) \xrightarrow{d_h} \cdots \xrightarrow{d_h} \Lambda_h^n(\Omega) \longrightarrow 0$$

Ref: D. Arnold, R. Falk, R. Winther, ***Finite element exterior calculus, homological techniques, and applications***, *Acta Numerica*, 1–155, 2006.

Outline

1. Prior Work on Functional Constraints
2. Prior Work on Stability
3. A Unifying Framework: Geometric PDEs

Types of Stability

Three notions of a "stable" PDE solution method:

1. **Well-posed stability:** Stable in the sense of being a well-posed PDE problem.
2. **Discrete stability:** Stable in the projection to a discrete problem.
3. **Numerical stability:** Stable numerically in a particular implementation.

We are most interested in discrete stability, i.e. the criteria required to conclude that the selected solution function depends continously on the input data.

Babuska's Theorem

Let U_h be our trial spaces (i.e. the spaces where we look for solutions) and V_h our test spaces (i.e. the spaces which we test against in the variational form). Let b_h be a family of continous bilinear forms:

$$\begin{cases} b_h(u, v_h), & u \in U, v \in V_h \\ |b_h(u, v_h)| \leq ||b_h|| ||u|| ||v_h|| \end{cases}$$

Suppose we want to solve the projection problem:

$$\begin{cases} u_h \in U_h \\ b_h(u_h, v_h) = b_h(u, v_h) \end{cases} \qquad \left(c.f. \int_\Omega \text{div}\, \sigma v\, dx = \int_\Omega fv\, dx, \quad \forall v \in L^2(\Omega) \right)$$

Suppose the discrete *inf-sup* condition holds:

$$\inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{|b_h(u_h, v_h)|}{||u_h|| ||v_h||} =: \gamma_h > 0$$

Then the problem has a unique solution u_h depending continously on u and

$$||u - u_h|| \leq \left(1 + \frac{||b_h||}{\gamma_h} \right) \inf_{w_h \in U_h} ||u - w_h||$$

Ref: Notation from L. Demkowicz, ***Babuska iff Brezzi ??***, *ICES Tech Report*, 2006.

Outline

Geometric PDEs

A method for smooth (molecular) surface construction:

Given $g(x) \geq 0$ on $\Omega \subset \mathbb{R}^3$, find a surface $\Gamma \subset \Omega$ minimizing the energy functional

$$E(\Gamma) = \int_\Gamma g(x)dx + \epsilon \int_\Gamma h(x,n)dx$$

where (x,n) is a surface point x with normal vector n and $h(x,n) \geq 0$. The search is done over a set of functions ϕ in some finite dimensional space S.

Goals:

- Cast this into a variational form and determine the natural basis for S so that the method is stable.
- Incorporate topological constraints (such as multiple components or a particular genus) into the bases for S.
- Analyze the sensitivity of the method under various choices of g, h, and ϵ.

Thank You

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http://www.math.utexas.edu/users/agillette/