Generalized Barycentric Coordinates for Polygonal Finite Elements

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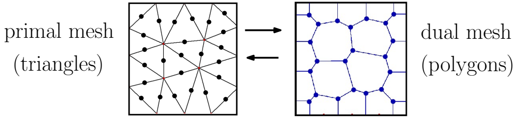
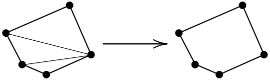
Why consider polygonal finite elements?

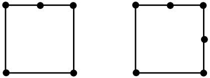
- **Theoretical:** Discrete Exterior Calculus considerations



- **Applied:** A new approach to longstanding meshing problems
  - Sliver removal by local remeshing



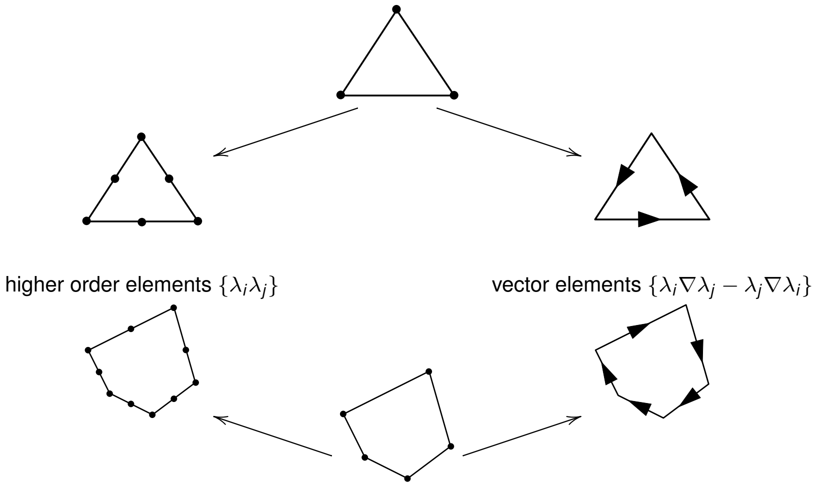
  - Canonical adaptive meshing elements



- **Practical:** Generic approach to coding would encompass old and new methods.

Overview of Approach

linear elements: $\{\lambda_i\}$ = (triangular) barycentric coordinates



linear elements: $\{\lambda_i\}$ = *generalized* barycentric coordinates

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1. Types of Generalized Barycentric Coordinates
2. Linear Elements
3. Quadratic ‘Serendipity’ Elements
4. Vector Elements

Definition

Let Ω be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on Ω if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on Ω.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision:** $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

Theorem [Warren, 2003]

If the λ_i are rational functions of degree $n - 2$, then they are unique.

Many generalizations to choose from . . .

- Wachspress
  ⇒ WACHSPRESS, *A Rational Finite Element Basis*, 1975.
- Sibson
  ⇒ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.
- Harmonic
  ⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN,
  Barycentric coordinates for convex sets, 2007.
- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.

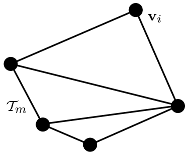
Many more in graphics contexts...

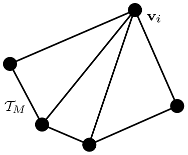
Triangulation Coordinates

Let $\mathcal{T}$ be a triangulation of Ω formed by adding edges between the $\mathbf{v}_j$ in some fashion. Define

$$\lambda_{i,\mathcal{T}}^{\text{Tri}} : \Omega \to \mathbb{R}$$

to be the barycentric function associated to $\mathbf{v}_i$ on triangles in $\mathcal{T}$ containing $\mathbf{v}_i$ and identically 0 otherwise. Trivially, these are barycentric coordinates on Ω.





Theorem [Floater, Hormann, Kós, 2006]

For a fixed i, let $\mathcal{T}_m$ denote any triangulation with an edge between $\mathbf{v}_{i-1}$ and $\mathbf{v}_{i+1}$. Let $\mathcal{T}_M$ denote the triangulation formed by connecting $\mathbf{v}_i$ to all the other $\mathbf{v}_j$. Any barycentric coordinate function λ_i satisfies the bounds

$$0 \leq \lambda_{i,\mathcal{T}_m}^{\text{Tri}}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_{i,\mathcal{T}_M}^{\text{Tri}}(\mathbf{x}) \leq 1, \quad \forall \mathbf{x} \in \Omega.$$

Harmonic Coordinates

Let $g_i : \partial\Omega \to \mathbb{R}$ be the piecewise linear function satisfying

$$g_i(\mathbf{v}_j) = \delta_{ij}, \quad g_i \text{ linear on each edge of } \Omega.$$

The **harmonic coordinate** function λ_i^{Har} is defined to be the solution of Laplace's equations with g_i as boundary data,

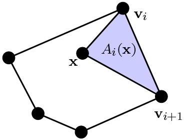
$$\begin{cases} \Delta\left(\lambda_i^{\text{Har}}\right) & = & 0, & \text{on } \Omega, \\ \lambda_i^{\text{Har}} & = & g_i. & \text{on } \partial\Omega. \end{cases}$$

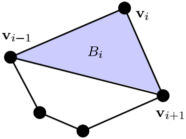
These coordinates are **optimal** in the sense that they minimize the norm of the gradient over all functions satisfying the boundary conditions:

$$\lambda_i^{\text{Har}} = \text{argmin}\left\{|\lambda|_{H^1(\Omega)} \ : \ \lambda = g_i \text{ on } \partial\Omega\right\}.$$

Wachspress Coordinates

Let $\mathbf{x} \in \Omega$ and define $A_i(\mathbf{x})$ and B_i as the areas shown.





Define the Wachspress weight function as

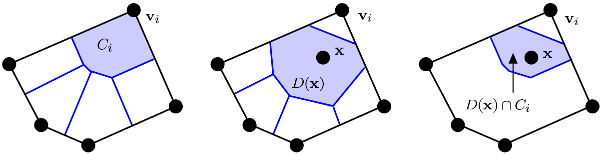
$$w_i^{\mathrm{Wach}}(\mathbf{x}) = B_i \prod_{j \neq i, i-1} A_j(\mathbf{x}).$$

The Wachspress coordinates are then given by the *rational* functions

$$\lambda_i^{\mathrm{Wach}}(\mathbf{x}) = \frac{w_i^{\mathrm{Wach}}(\mathbf{x})}{\sum_{j=1}^{n} w_j^{\mathrm{Wach}}(\mathbf{x})}$$

Sibson (Natural Neighbor) Coordinates

Let P denote the set of vertices $\{\mathbf{v}_i\}$ and define $P' = P \cup \{\mathbf{x}\}$.



$$\begin{aligned} C_i &:= |V_P(\mathbf{v}_i)| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{v}_i| < |\mathbf{y} - \mathbf{v}_j| \ , \forall j \neq i\}| \\ &= \text{area of cell for } \mathbf{v}_i \text{ in Voronoi diagram on the points of } P, \end{aligned}$$

$$\begin{aligned} D(\mathbf{x}) &:= |V_{P'}(\mathbf{x})| = |\{\mathbf{y} \in \Omega : |\mathbf{y} - \mathbf{x}| < |\mathbf{y} - \mathbf{v}_i| \ , \forall i\}| \\ &= \text{area of cell for } \mathbf{x} \text{ in Voronoi diagram on the points of } P'. \end{aligned}$$

By a slight abuse of notation, we also define

$$D(\mathbf{x}) \cap C_i := |V_{P'}(\mathbf{x}) \cap V_P(\mathbf{v}_i)|.$$

The Sibson coordinates are defined to be

$$\lambda_i^{\text{Sibs}}(\mathbf{x}) := \frac{D(\mathbf{x}) \cap C_i}{D(\mathbf{x})} \qquad \text{or, equivalently,} \qquad \lambda_i^{\text{Sibs}}(\mathbf{x}) = \frac{D(\mathbf{x}) \cap C_i}{\sum_{j=1}^{n} D_j(\mathbf{x}) \cap C_j}.$$

Outline

Optimal Convergence Estimates on Polygons

Let Ω be a convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

For linear elements, an **optimal convergence estimate** has the form

$$\underbrace{\left|\left| u - \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C \operatorname{diam}(\Omega) \, |u|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega). \tag{1}$$

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in H^1 norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an ***H^1*-interpolant estimate** over domains of diameter one:

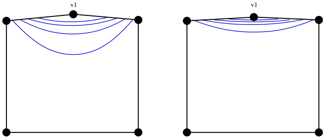
$$\left|\left| \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)} \leq C_I \, ||u||_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2}$$

For (2), it suffices to **bound the gradients** of the $\{\lambda_i\}$, i.e. prove $\exists\, C_\lambda \in \mathbb{R}$ such that

$$||\nabla \lambda_i||_{L^2(\Omega)} \leq C_\lambda. \tag{3}$$

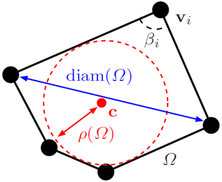
Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need estimates of the geometry.



Let $\rho(\Omega)$ denote the radius of the largest inscribed circle.
The **aspect ratio** γ is defined by

$$\gamma = \frac{\text{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh

- **G1.** ***Bounded aspect ratio:*** There exists $\gamma^* < \infty$ such that
$$\gamma < \gamma^*$$
- **G2.** ***Minimum edge length:*** There exists $d_* > 0$ such that
$$|\mathbf{v}_i - \mathbf{v}_{i-1}| > d_*$$
- **G3.** ***Maximum interior angle:*** There exists $\beta^* < \pi$ such that
$$\beta_i < \beta^*$$

Summary of convergence results

Theorem

In the table below, any necessary geometric criteria to achieve the optimal convergence estimate are denoted by N. The set of geometric criteria denoted by S in each row are sufficient to guarantee estimate.

GILLETTE, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation*, Advances in Computational Mathematics, accepted, 2011.

		G1 aspect ratio	G2 min. edge	G3 max angle
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Harmonic	λ^{Har}	S	-	-

Implication of convergence results

restrictive — GEOMETRIC CONSTRAINTS — flexible

Triangulate — Wachspress — Sibson — Harmonic

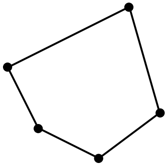
easy — EFFICIENT IMPLEMENTATION — difficult

Triangulate — Wachspress — Sibson — Harmonic

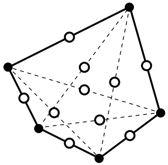
Outline

From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



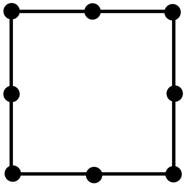
Why is this naïve?

- For an n-gon, this construction gives $n + \binom{n}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only 2n functions needed!*

Problem Statement

Construct $2n$ basis functions associated to the vertices and edge midpoints of an arbitrary n-gon such that a quadratic convergence estimate is obtained.

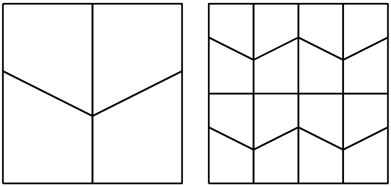
Prior work - Quadrilateral serendipity elements



For quadrilaterals, the 'serendipity' element for **rectangles** has long been known to provide quadratic convergence.

STRANG, FIX, *An analysis of the finite element method*, 1973.
HUGHES, *The finite element method*, 1987.

The technique works more generally for **affine** mappings of the reference element to a physical element *('affine' = preserves collinearity and ratios of distances)*



$n = 2$ $n = 4$

For **non-affine** meshes of quadrilaterals, however, the serendipity construction is known to provide sub-optimal convergence.

ARNOLD, BOFFI, FALK, *Approximation by Quadrilateral Finite Elements*, Mathematics of Computation, 2002.

Failure for non-affine reference element mappings

Mapped biquadratic elements

	square meshes						trapezoidal meshes					
	$\|u - u_h\|_{L^2}$			$\|\nabla(u - u_h)\|_{L^2}$			$\|u - u_h\|_{L^2}$			$\|\nabla(u - u_h)\|_{L^2}$		
n	err.	%	rate	err.	%	rate	err.	%	rate	err.	%	rate
2	3.5e−02	2.877		4.5e−01	37.253		4.8e−02	3.951		5.9e−01	48.576	
4	4.4e−03	0.360	3.0	1.1e−01	9.333	2.0	5.8e−03	0.475	3.1	1.5e−01	12.082	2.0
8	5.5e−04	0.045	3.0	2.8e−02	2.329	2.0	7.1e−04	0.058	3.0	3.7e−02	3.017	2.0
16	6.9e−05	0.006	3.0	7.1e−03	0.583	2.0	8.7e−05	0.007	3.0	9.2e−03	0.753	2.0
32	8.6e−06	0.001	3.0	1.8e−03	0.146	2.0	1.1e−05	0.001	3.0	2.3e−03	0.188	2.0
64	1.1e−06	0.000	3.0	4.4e−04	0.036	2.0	1.3e−06	0.000	3.0	5.7e−04	0.047	2.0

Serendipity elements

	square meshes						trapezoidal meshes					
	$\|u - u_h\|_{L^2}$			$\|\nabla(u - u_h)\|_{L^2}$			$\|u - u_h\|_{L^2}$			$\|\nabla(u - u_h)\|_{L^2}$		
n	err.	%	rate	err.	%	rate	err.	%	rate	err.	%	rate
2	3.5e−02	2.877		4.5e−01	37.252		5.0e−02	4.066		6.2e−01	51.214	
4	4.4e−03	0.360	3.0	1.1e−01	9.333	2.0	6.7e−03	0.548	2.9	1.8e−01	14.718	1.8
8	5.5e−04	0.045	3.0	2.8e−02	2.329	2.0	9.7e−04	0.080	2.8	5.9e−02	4.836	1.6
16	6.9e−05	0.006	3.0	7.1e−03	0.583	2.0	1.6e−04	0.013	2.6	2.3e−02	1.890	1.4
32	8.6e−06	0.001	3.0	1.8e−03	0.146	2.0	3.3e−05	0.003	2.3	1.0e−02	0.842	1.2
64	1.1e−06	0.000	3.0	4.4e−04	0.036	2.0	7.4e−06	0.001	2.1	4.9e−03	0.401	1.1

ARNOLD, BOFFI, FALK, *Approximation by Quadrilateral Finite Elements*, 2002.

Generalized barycentric quadrilateral elements

- Generalized barycentric coordinates allow for a quadratic serendipity construction on **any** quadrilateral.
- Since the analysis holds for affine mappings, these serve as reference elements for a wider range of quadrilaterals.
- The trapezoidal meshes satisfy the geometry bounds and hence we can recover the optimal convergence rate.

	$\|\|u - u_h\|\|_{L^2}$		$\|\|\nabla(u - u_h)\|\|_{L^2}$	
n	error	rate	error	rate
2	2.34e-3		2.22e-2	
4	3.03e-4	2.95	6.10e-3	1.87
8	3.87e-5	2.97	1.59e-3	1.94
16	4.88e-6	2.99	4.04e-4	1.97
32	6.13e-7	3.00	1.02e-4	1.99
64	7.67e-8	3.00	2.56e-5	1.99
128	9.59e-9	3.00	6.40e-6	2.00
256	1.20e-9	3.00	1.64e-6	1.96

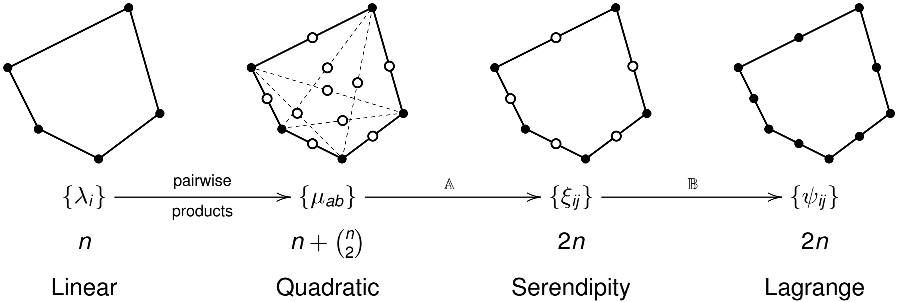
RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot = Lagrangian domain point
= all functions in the set evaluate to 0
except the associated function which evaluates to 1

open dot = non-Lagrangian domain point
= partition of unity satisfied, but not Lagrange property



From quadratic to serendipity

Serendipity basis functions ξ_{ij} are constructed as a linear combination of pairwise product functions μ_{ab}:

$$[\xi_{ij}] = \mathbb{A} \begin{bmatrix} \mu_{aa} \\ \hline \mu_{a(a+1)} \\ \hline \mu_{ab} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & c_{ab}^{ij} \end{bmatrix} \begin{bmatrix} \mu_{aa} \\ \hline \mu_{a(a+1)} \\ \hline \mu_{ab} \end{bmatrix}$$

The quadratic basis is ordered as follows:

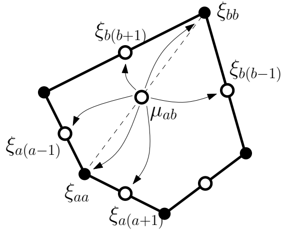
$$\begin{aligned} \mu_{aa} &= \text{basis functions associated with vertices} \\ \mu_{a(a+1)} &= \text{basis functions associated with edge midpoints} \\ \mu_{ab} &= \text{basis functions associated with interior diagonals,} \\ &\quad\ \text{i.e. } b \notin \{a-1, a, a+1\} \end{aligned}$$

- The first two types are left alone, resulting in the identify matrix above.
- The c_{ab}^{ij} values define how the interior basis functions are added into the boundary basis functions.

From quadratic to serendipity

We *require* the serendipity basis to have quadratic approximation power:

- **Constant precision (CP)**: $\sum_i \xi_{ii} + 2\xi_{i(i+1)} = 1.$
- **Linear precision (LP)**: $\sum_i \mathbf{v}_i \xi_{ii} + 2\mathbf{v}_{i(i+1)}\xi_{i(i+1)} = \mathbf{x}.$
- **Quadratic precision (QP)**: $\sum_i \mathbf{v}_i \mathbf{v}_i^T \xi_{ii} + (\mathbf{v}_i \mathbf{v}_{i+1}^T + \mathbf{v}_{i+1}\mathbf{v}_i^T)\xi_{i(i+1)} = \mathbf{x}\mathbf{x}^T.$



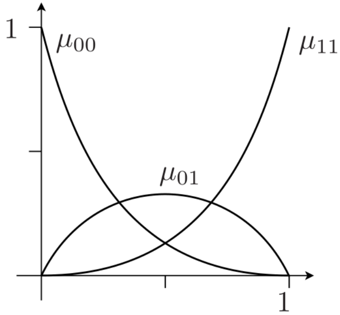
- Six constraints (CP, LP, QP) $\Rightarrow$ six non-zero c_{ab}^{ij} per column.
- We select (arbitrarily) that μ_{ab} contributes to $\xi_{a,a}$, $\xi_{b,b}$, and their neighbors.

Theorem

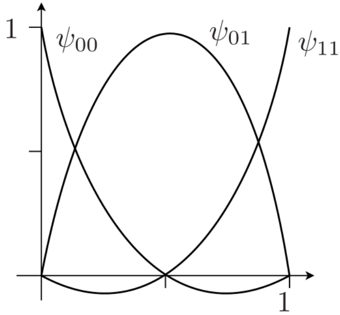
Constants $\{c_{ij}^{ab}\}$ exist for any convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies the CP, LP, and QP requirements.

Pairwise products vs. Lagrange basis

Pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:

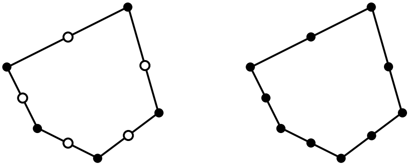


Pairwise products of barycentric functions



Lagrange basis

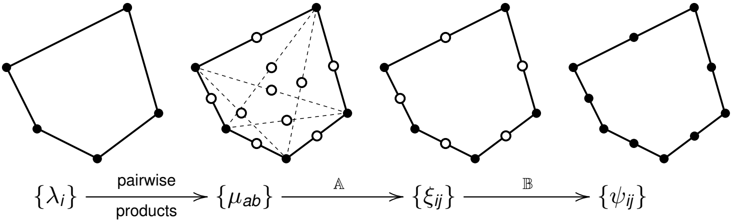
Translation between these two bases is straightforward and generalizes to the higher dimensional case...



$$\{\xi_{ij}\} \xrightarrow{\quad\mathbb{B}\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{1n} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{1n} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



Theorem

Given bounds on polygon aspect ratio (G1), minimum edge length (G2), and maximum interior angles (G3):

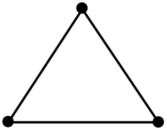
- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- The basis $\{\psi_{ij}\}$ interpolates smooth data with $O(h^2)$ error.

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

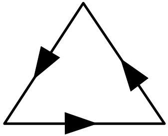
Outline

From scalar to vector elements

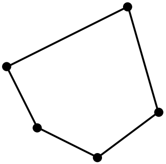
Barycentric functions are used to define H(curl) vector elements on triangles:



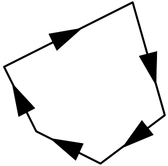
$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$



Generalized barycentric functions provide H(curl) elements on polygons:



$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a \nabla \lambda_b - \lambda_b \nabla \lambda_a\}$$



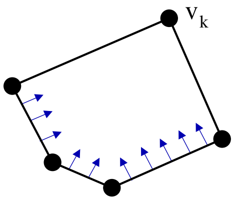
This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications.

GILLETTE, BAJAJ *Dual Formulations of Mixed Finite Element Methods with Applications*
Computer-Aided Design 43:10, pages 1213-1221, 2011.

Conformity and interpolation properties

Conformity: The basis functions $\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\}$ interpolate an $H(\text{curl})$ function.

Let $T_E \vec{v}$ denote the tangential projection of $\vec{v}$ to an edge E.



$$H(\text{curl }) := \left\{ \vec{v} \in \left(L^2(\Omega)\right)^3 \quad \text{s.t.} \quad \nabla \times \vec{v} \in \left(L^2(\Omega)\right)^3 \right\}$$

$$\vec{v} \in H(\text{curl }) \iff T_E \vec{v} \in C^0, \quad \forall \text{ edges } E \text{ in mesh}$$

$$\lambda_k \equiv 0 \quad \text{on } E \not\ni v_k$$

$$\therefore \nabla \lambda_k \perp E \quad \text{on } E \not\ni v_k$$

$$\therefore \ T_E(\lambda_i \nabla \lambda_j) \neq 0 \iff \mathbf{v}_i, \mathbf{v}_j \in E$$

Interpolation: The basis functions are Lagrange-like for edge integrals.

$$T_{\vec{e_{ij}}}(\nabla \lambda_i) = \frac{1}{|e_{ij}|}, \quad \text{since the } \lambda_i \text{ are linear on edges.}$$

$$\int_{e_{ij}} (\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i) \cdot \vec{e_{ij}} = \frac{1}{|e_{ij}|} \int_{e_{ij}} \lambda_i + \lambda_j = \frac{1}{|e_{ij}|} \int_{e_{ij}} 1 = 1.$$

Future directions

Future work and open problems

- Extension to 3D generalized barycentric functions.
- Extension to 3D vector interpolation functions on polytopes.
- Implementation in a finite element solver for comparison studies.

Questions?

- Slides and pre-prints available at http://ccom.ucsd.edu/∼agillette