Polytopal Finite Element Methods: Theory and Application

Andrew Gillette
Department of Mathematics
University of Arizona
What are finite element methods?

The **finite element method** is a way to numerically approximate the solution to PDEs.

\[ \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix} \]

**CHARACTERIZE**
Real analysis
PDEs

**DISCRETIZE**
Geometry & Topology
Combinatorics

**SOLVE**
Linear algebra
Numerical analysis

In this talk:
- Why use meshes of polygons / polyhedra for discretization?
- What are some finite element methods that allow such meshes?
- Where is this line of research headed?
1. Why use polytopal meshes?
2. What are some polyhedral finite element methods?
3. Where is this line of research headed?
Outline

1. Why use polytopal meshes?

2. What are some polyhedral finite element methods?

3. Where is this line of research headed?
Domain meshing with simplices or cubes is now so well-understood that there is a **Periodic Table of Finite Elements:**

→ Viewable online at [femtable.org](http://femtable.org)
→ Scientific content prepared by Doug Arnold and Anders Logg
Tetrahedral volume mesh for CFD, using DistMesh software.
(courtesy of Per-Olof Persson)
Tetrahedral volume mesh for CFD, using Pointwise software.

(from blog.pointwise.com)
Hybrid hex / pyramid / prism / tet mesh for CFD, using ITI Transcendata software.
(from a keynote address at Geometric Modeling and Processing 2015)
Body-aligned prismatic polyhedral meshes for CFD, using CD-adapco software.
(from cd-adapco.com image gallery)
Volume meshing for Computational Fluid Dynamics

Polyhedral mesh of a Bell 407 helicopter and surrounding volume.
(from cd-adapco.com image gallery)
Volume meshing for... cell phone design!

A polyhedral mesh used to study heat transfer and cooling of a cell phone.  
(from cd-adapco.com image gallery)
A polyhedral mesh conforming to a surface triangulation using *VoroCrust* software.
(from Scott Mitchell, Sandia National Labs)
A polyhedral mesh conforming to a surface triangulation using VoroCrust software. (from Scott Mitchell, Sandia National Labs)
The requisite Stanford Bunny example using **VoroCrust** software.
(from Scott Mitchell, Sandia National Labs)
Hexahedral meshing is polyhedral meshing

Meshes of generic hexahedra require a generalized theory of polyhedral discretization, related to but distinct from the theory for perfect tensor product meshes.

↑ Heart mesh made using Continuity software, National Biomedical Computation Resource, UCSD

← Hole mesh made using CUBIT Geometry and Mesh Generation Toolkit, Sandia National Labs
Elasticity modeling

Standard triangular FEM cannot model maximal stretch factors due to numerical errors from the deformation.

Elasticity modeling

The flexibility of polyhedral meshes allows greater shape deformation and more realistic stretch factors.

$\lambda = 2.1$

Chi et al. “Polygonal finite elements for finite elasticity.”
Talischi et al. “Gradient correction for polygonal and polyhedral finite elements.”
Outline

1. Why use polytopal meshes?
2. What are some polyhedral finite element methods?
3. Where is this line of research headed?
The Finite Element Method for Poisson’s Problem

**Continuous problem:** find \( u \in U \) s.t.
\[
\Delta u = f \quad \text{on } \Omega \subset \mathbb{R}^n
\]

**Weak form:** find \( u \in U \) s.t.
\[
\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} f \, v, \quad \forall v \in V \quad (\text{dim } V = \infty)
\]

**Discrete form:** find \( u_h \in U_h \) s.t.
\[
\int_{\Omega} \nabla u_h \cdot \nabla v_h = \int_{\Omega} f \, v_h, \quad \forall v_h \in V_h \quad (\text{dim } V_h < \infty)
\]

**Linear system:** Set \( U_h := V_h \) (Galerkin method). Find \( u \in \mathbb{R}^{\text{dim } V_h} \) solving
\[
[ K ]_{ji} [ u ]_i = [ f ]_j, \quad \forall v_j \in \text{basis for } V_h
\]

where
\[
[ K ]_{ji} = \int_E \nabla v_i \cdot \nabla v_j \quad \text{and} \quad [ f ]_j = \int_E f(x) v_j(x)
\]

**Key challenge when \( E \) is a polygon/polyhedron:**
Efficient computation or approximation of \( \int_E \nabla v_i \cdot \nabla v_j, \quad \forall v_j \in \text{basis for } V_h \)
A few kinds of polytopal element methods.

CDO = Compatible discrete operator schemes
A few kinds of polytopal element methods... 

CDO  =  Compatible discrete operator schemes  
DEC  =  Discrete exterior calculus
A few kinds of polytopal element methods...

CDO = Compatible discrete operator schemes
DEC = Discrete exterior calculus
DPG = Discontinuous Petrov-Galerkin
A few kinds of polytopal element methods . . .

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
</tbody>
</table>

Moreover, some kinds of methods are the same as other methods.

A few kinds of polytopal element methods...

- **CDO** = Compatible discrete operator schemes
- **DEC** = Discrete exterior calculus
- **DPG** = Discontinuous Petrov-Galerkin
- **FES** = Finite element systems
- **GBC** = Generalized barycentric coordinate methods

Moreover, some kinds of methods are the same as methods...

"Polytopal Element Methods in Mathematics and Engineering"

- Special NSF-funded workshop held at Georgia Tech in Oct 2015
- Slides from talks: [http://www.poems15.gatech.edu/](http://www.poems15.gatech.edu/)
A few kinds of polytopal element methods...

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
<tr>
<td>GBC</td>
<td>Generalized barycentric coordinate methods</td>
</tr>
<tr>
<td>GS</td>
<td>Gradient schemes</td>
</tr>
</tbody>
</table>

Moreover, some kinds of methods are the same as other methods.

A few kinds of polytopal element methods...

CDO = Compatible discrete operator schemes
DEC = Discrete exterior calculus
DPG = Discontinuous Petrov-Galerkin
FES = Finite element systems
GBC = Generalized barycentric coordinate methods
GS = Gradient schemes
HFV = Hybrid finite volume methods
A few kinds of polytopal element methods...

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Method Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
<tr>
<td>GBC</td>
<td>Generalized barycentric coordinate methods</td>
</tr>
<tr>
<td>GS</td>
<td>Gradient schemes</td>
</tr>
<tr>
<td>HFV</td>
<td>Hybrid finite volume methods</td>
</tr>
<tr>
<td>HDG</td>
<td>Hybrid discontinuous Galerkin methods</td>
</tr>
</tbody>
</table>
A few kinds of polytopal element methods...

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
<tr>
<td>GBC</td>
<td>Generalized barycentric coordinate methods</td>
</tr>
<tr>
<td>GS</td>
<td>Gradient schemes</td>
</tr>
<tr>
<td>HFV</td>
<td>Hybrid finite volume methods</td>
</tr>
<tr>
<td>HDG</td>
<td>Hybrid discontinuous Galerkin methods</td>
</tr>
<tr>
<td>HHO</td>
<td>Hybrid higher-order methods</td>
</tr>
</tbody>
</table>
A few kinds of polytopal element methods...

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
<tr>
<td>GBC</td>
<td>Generalized barycentric coordinate methods</td>
</tr>
<tr>
<td>GS</td>
<td>Gradient schemes</td>
</tr>
<tr>
<td>HFV</td>
<td>Hybrid finite volume methods</td>
</tr>
<tr>
<td>HDG</td>
<td>Hybrid discontinuous Galerkin methods</td>
</tr>
<tr>
<td>HHO</td>
<td>Hybrid higher-order methods</td>
</tr>
<tr>
<td>MFD</td>
<td>Mimetic finite difference</td>
</tr>
</tbody>
</table>
A few kinds of polytopal element methods...

CDO = Compatible discrete operator schemes
DEC = Discrete exterior calculus
DPG = Discontinuous Petrov-Galerkin
FES = Finite element systems
GBC = Generalized barycentric coordinate methods
GS = Gradient schemes
HFV = Hybrid finite volume methods
HDG = Hybrid discontinuous Galerkin methods
HHO = Hybrid higher-order methods
MFD = Mimetic finite difference
VEM = Virtual element methods
## A few kinds of polytopal element methods...

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDO</td>
<td>Compatible discrete operator schemes</td>
</tr>
<tr>
<td>DEC</td>
<td>Discrete exterior calculus</td>
</tr>
<tr>
<td>DPG</td>
<td>Discontinuous Petrov-Galerkin</td>
</tr>
<tr>
<td>FES</td>
<td>Finite element systems</td>
</tr>
<tr>
<td>GBC</td>
<td>Generalized barycentric coordinate methods</td>
</tr>
<tr>
<td>GS</td>
<td>Gradient schemes</td>
</tr>
<tr>
<td>HFV</td>
<td>Hybrid finite volume methods</td>
</tr>
<tr>
<td>HDG</td>
<td>Hybrid discontinuous Galerkin methods</td>
</tr>
<tr>
<td>HHO</td>
<td>Hybrid higher-order methods</td>
</tr>
<tr>
<td>MFD</td>
<td>Mimetic finite difference</td>
</tr>
<tr>
<td>VEM</td>
<td>Virtual element methods</td>
</tr>
<tr>
<td>WG</td>
<td>Weak Galerkin methods</td>
</tr>
</tbody>
</table>
A few kinds of polytopal element methods...

- CDO = Compatible discrete operator schemes
- DEC = Discrete exterior calculus
- DPG = Discontinuous Petrov-Galerkin
- FES = Finite element systems
- GBC = Generalized barycentric coordinate methods
- GS = Gradient schemes
- HFV = Hybrid finite volume methods
- HDG = Hybrid discontinuous Galerkin methods
- HHO = Hybrid higher-order methods
- MFD = Mimetic finite difference
- VEM = Virtual element methods
- WG = Weak Galerkin methods

Moreover, some kinds of *** methods are the same as *** methods...
A few kinds of polytopal element methods... 

CDO = Compatible discrete operator schemes
DEC = Discrete exterior calculus
DPG = Discontinuous Petrov-Galerkin
FES = Finite element systems
GBC = Generalized barycentric coordinate methods
GS = Gradient schemes
HFV = Hybrid finite volume methods
HDG = Hybrid discontinuous Galerkin methods
HHO = Hybrid higher-order methods
MFD = Mimetic finite difference
VEM = Virtual element methods
WG = Weak Galerkin methods

Moreover, some kinds of ⋆ ⋆ ⋆ methods are the same as ⋆ ⋆ ⋆ methods...

“Polytopal Element Methods in Mathematics and Engineering”

→ Special NSF-funded workshop held at Georgia Tech in Oct 2015
→ Slides from talks: http://www.poems15.gatech.edu/
Outline

1. Why use polytopal meshes?
2. What are some polyhedral finite element methods?
3. Where is this line of research headed?
Robust software for polygonal meshing is freely available... but robust **polyhedral meshing** is still in development.

Industry and engineering researchers are interested in polytopal methods... but **increased communication** will be crucial for widespread adoption.

There is no ‘silver bullet’ method... so there are **lots of open questions**!

My main interests are:

- finite element theory
- polyhedral elements
- biomedical applications
Acknowledgments

Funding

My research is supported in part by a grant from the National Science Foundation.

Collaborators

- Chandrajit Bajaj, UT Austin, computer science
- Snorre Christiansen, U. Oslo, math
- Michael Floater, U. Oslo, math
- Michael Holst, UC San Diego, math
- Peter Kekenes-Huskey, U. Kentucky, chemistry
- Alexander Rand, CD-adapco, industry
- N. Sukumar, UC Davis, civil engineering
- Kevin Vincent, UC San Diego, bioengineering
- Yunrong Zhu, Idaho State, math

Slides and pre-prints: http://math.arizona.edu/~agillette/