# Geometric Discretization of Stress and Elasticity

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# Outline

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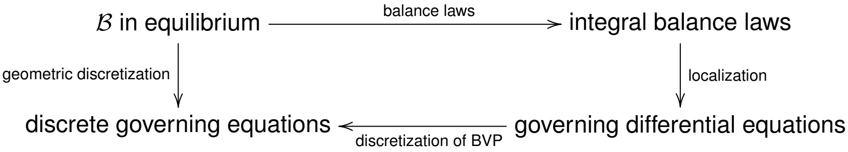
# Why Geometric Discretization?




Geometric discretization techniques attempt to simplify the process of modeling the balance laws discretely. For field theory modeling, this first requires a full comprehension of the geometry of the continuous theory.

# Why Differential Forms?

- **Generality:** Differential forms provide a unified abstract setting to correctly capture the full mathematical structures of the field geometry in question.
- **Coordinate Independence:** Forms do not rely on the particular embeddings of the surfaces being modeled. Coordinates are specified at the last stage of modeling, thereby enhancing numerical computations.
- **Dependence Identification:** Casting problems into a differential forms framework highlights whether the involved quantities and operators are topologically or metric dependent. This allows for better design of discrete algorithms that preserve the continuous physical laws being modeled.

# Comparison to Prior Work

Yavari is careful to point out that geometric discretization is ***not*** the same as an approach called the "cell method" developed by Tonti* and expanded by others (Cosmi; Ferretti; Pani et al.).

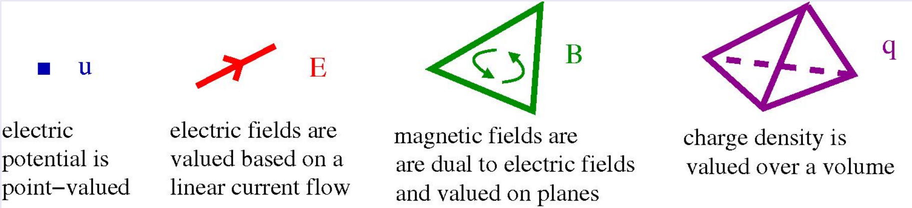
- Cell Method: discretize the domain (i.e. make a mesh) and assume that deformation is homogeneous within each cell. This formulation is called *ab initio* since it does not reference the corresponding continuum formulation.
- Geometric Discretization method: discretize the operators (i.e. use discrete differential forms) and deduce the according relationships on a mesh of the domain. This approach to stress and elasticity is quite new, dating back only to 2006.

*E. Tonti *A direct discrete formulation of field laws: The cell method*, Comput. Model. Eng. Sci, 2001.

# Review: Forms for EM

## Characterization of *k*-forms

A *k*-form represents an intrinsically *k*-dimensional phenomena and can be integrated over a *k*-dimensional region.

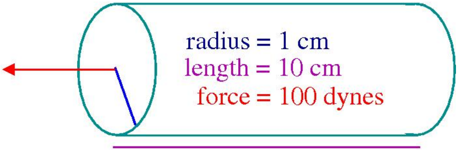



$$\left.\begin{aligned} \nabla \times H &= \frac{\partial D}{\partial t} + J_E \\ \nabla \times E &= -\frac{\partial B}{\partial t} \\ \nabla \cdot B &= 0 \\ \nabla \cdot D &= \rho_E \end{aligned}\right\} \text{vector calculus vs. differential forms} \left\{\begin{aligned} \mathbf{d}H &= \frac{\partial D}{\partial t} + J_E \\ \mathbf{d}E &= -\frac{\partial B}{\partial t} \\ \mathbf{d}B &= 0 \\ \mathbf{d}D &= \rho_E \end{aligned}\right.$$

# Outline

# Stress, Strain, and Elasticity

Consider a piece of clay as shown:




- The **stress** on the clay is a measure of (force / area):

$$\text{stress} = \frac{100}{\pi} \frac{\text{dynes}}{\text{cm}^2}$$

  It has a direction parallel to the force (normal to cross-section).
- The **strain** on the clay is the fractional extension in the direction of the stress. If the force caused the clay to stretch 2 cm, the strain experienced was 2 cm / 10 cm = 0.2, a dimensionless quantity.
- The **elasticity** of the clay is its ability (or inability) to return to its original shape after receiving stress.

Hence, before tackling the geometry of elasticity, we must first understand the geometry of stress.

# Classical Stress Theory

The classical approach to modeling stress in $\mathbb{R}^3$ is as follows.

1. Assume the existence of a **traction** vector field

$$\mathbf{t}(\vec{x}, t; \vec{n}) : \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

   which gives the force per unit area exerted on a surface through $\vec{x}$ with normal $\vec{n}$ at time $t$.

   sample $\mathbf{t}$ unit: $\frac{kg}{ms^2}$

2. Suppose the surface is moving according to a spatial velocity field $\vec{v}(\vec{x}, t)$. Then the rate of work $R^t$ done by the traction forces on an oriented surface $S$ is

$$R^t = \int_S < \vec{v}, \mathbf{t} > da$$

   Throughout, $< \cdot, \cdot >$ indicates the “natural pairing” of the fields involved, which in this case means the dot product.

   sample $R^t$ unit: $\frac{kgm^2}{s^3}$

# Classical Stress Theory

3. Cauchy's Theorem, a.k.a. the Stress Principle states that **t** is linear in $\vec{n}$. That is, there exists a 2-tensor $\sigma$ such that

$$\mathbf{t}(\vec{x}, t, \vec{n}) = < \sigma(\vec{x}, t), \vec{n} >$$

To clarify: $\sigma$ is a 2-tensor dependent on $\vec{x}$ and $t$. Its inputs as a tensor are $\vec{v}$ and $\vec{n}$, in that order. Thus, $\sigma$ can be associated to a 3x3 matrix (whose coefficients depend on $\vec{x}$ and $t$) where the action of $\sigma$ on $\{\vec{v}, \vec{n}\}$ is

$$[v_1 \, v_2 \, v_3] \begin{bmatrix} \sigma_{11} & \cdots & \\ \vdots & \ddots & \\ & & \sigma_{33} \end{bmatrix} [n_1 \, n_2 \, n_3]^T$$

The drawback to this approach: $\sigma$ is highly coordinate dependent.

# Outline

# Stress with Differential Forms

Solution: use differential forms to abstract the stress model.

- Since stress is dependent on both a 1-form (velocity vector field) and a 2-form (patch of a surface) it is properly modeled as a tensor product of forms:

$$\mathcal{T} \in \Lambda^1(\mathcal{R}) \otimes \Lambda^2(\mathcal{R})$$

  where $\Lambda^k(\mathcal{R})$ is the space of $k$-forms on our surface $\mathcal{R}$. We say that $\mathcal{T}$ is a covector-valued 2-form.

- In coordinates, we can write:

$$\mathcal{T} = \sigma_{ab}\mathbf{d}x^a \otimes (*\mathbf{d}x^b)$$

  To clarify, $\sigma_{ab}$ is the matrix associated to the stress tensor, $\mathbf{d}x^a$ indicates the appropraite coordinate of velocity $\vec{v}$, and $*\mathbf{d}x^b$ indicates the Hodge star applied to $\vec{n}$, i.e. the appropriate coordinates of the plane defined by $\vec{n}$.

- Kanso et al. explain "Physically, $\mathcal{T}$ can be interpreted as follows: the stress, upon pairing with a velocity field, provides an area-form that is ready to be integrated over a surface to give the rate of work done by the stress on that surface."

# Rate of Work with Differential Forms

Using differential forms, we re-write the rate of work $R_t$ as follows.

$$
\begin{aligned}
R^t &= \int_S < \vec{v}, \mathbf{t} > da &&= \int_S < \vec{v}, \sigma(\cdot, \vec{n}) > da \\
&= \int_S \sigma(\vec{v}, \vec{n}) da &&= \int_S < \sigma(\vec{v}, \cdot), \vec{n} da > \\
&= \int_S *_2\sigma(v, \cdot) \\
&= \int_S < \vec{v}, *_2\sigma > \\
&= \int_S < \vec{v}, \mathcal{T} >
\end{aligned}
$$

Kanso et al: “Notice that if the orientation of $S$ switches, then the sign of the integral automatically switches and this corresponds to the change of sign of $\vec{n}$ in the traditional approach.” In other words, we have removed the coordinate-dependence from the definition of stress.

# Discrete Stress

Given a simplicial complex $K$, a $k$-chain is a linear combination of the $k$-simplicies of $K$. Note that a triangulated surface mesh is a simplicial complex.

A discrete differential $k$-form is a $k$-cochain $\omega$, i.e. a linear map from the space of $k$-chains $\mathcal{C}_k$ to $\mathbb{R}$:

$$\omega : \mathcal{C}_k \longrightarrow \mathbb{R}$$

A discrete vector-valued differential $k$-form $\psi$ returns a vector instead of an element of $\mathbb{R}$:

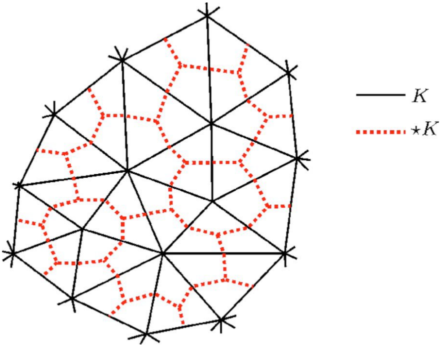
$$\psi : \mathcal{C}_k \longrightarrow \mathbb{V}, \quad \mathbb{V} = \mathbb{R}^3 \text{ or other vector space}$$

A discrete covector-valued $k$-form $\alpha$ returns a covector instead of a vector:

$$\alpha : \mathcal{C}_k \longrightarrow \mathbb{V}^*, \quad \text{i.e. } \alpha(c_k) \text{ is a vector-valued function}$$

Discrete stress should be a discrete covector-valued 2-form (for surfaces) since, given an area patch, it should output a method for evaluating the velocity vector field. To explain this further, we need to discuss primal and dual meshes.

# Primal and Dual Meshes




- A primal mesh $K$ has a dual mesh $\star K$, such as the barycentric dual shown.
- There is a 1-1 correspondence between primal $k$-cells and dual $(n-k)$-cells.
- However, the dual mesh lacks many basic properties of the primal mesh, e.g. convex cells and a fixed number of edges per cell.
- Further, it is non-canonical as other types of dual meshes (e.g. circumcentric) exist.

| | | |
|---|---|---|
| discrete primal $k$-form | $\longrightarrow$ | valued on primal $k$-cells |
| discrete dual $k$-form | $\longrightarrow$ | valued on dual $k$-cells |

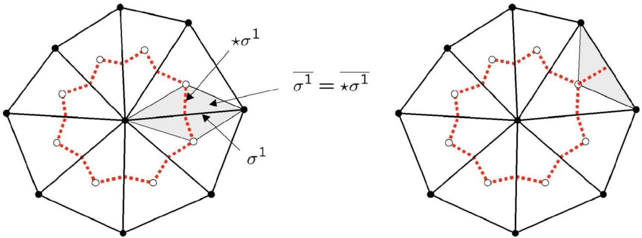
figure from [Yavari 2008]

# Discrete Stress

Discrete stress $\mathcal{T}_h$ is a discrete covector-valued dual 2-form.

$$\text{primal vertex } \sigma^0 \quad \xrightarrow{\star} \quad \text{dual 2-cell } \star\sigma^0 \quad \xrightarrow{\mathcal{T}_h} \quad \text{covector on } \partial(\star\sigma^0)$$

Further, the covector $\mathcal{T}_h(\star\sigma^0)$ has non-zero values *only* on the 1-cells $\partial(\star\sigma^0)$ and its evaluation depends on the orientation of $\partial(\star\sigma^0)$, as induced by the orientation of the surface.




Upshot of this approach: guarantees balance of linear momentum on dual 2-cells.

figure from [Yavari 2008]

# Balance of Linear Momentum

Differential forms generalize the various balance laws in elasticity theory. We will focus on only one example: **balance of linear momentum**. Fix the variables

$\sigma$ := Cauchy Stress 2-tensor

$\rho$ := mass density

$\vec{b}$ := body force

$\vec{a}$ := inertial force

The geometric version is

$$\mathrm{div}\sigma + \rho\vec{b} = \rho\vec{a}$$

The differential forms version is

$$\mathbf{d}\mathcal{T} + \mathbf{b} \otimes \rho = \mathbf{a} \otimes \rho$$

The geometric discretization is

$$< \mathbf{d}\mathcal{T}, \star c_k > + < \mathbf{b}, \star c_k > = < \mathbf{a}, \star c_k >$$

Note **a** and **b** are covector-valued 3-forms. Also note that the exterior derivative **d** is acting on a covector-valued form $\mathcal{T}$; the specification of this operation requires some additional theory beyond the scope of this talk.

# Geometric Formulation of Linear Elasticity

For problems of dimension $p$=2 or 3 we have these matrices:

| Name | Symbol | Size |
|---|---|---|
| $k$th primal incidence matrix ($\partial_k^T$) | $\mathbb{M}_k$ | $\#\sigma^{k+1} \times \#\sigma^k$ |
| $k$th dual incidence matrix | $\widetilde{\mathbb{M}}_k$ | $\# \star \sigma^{p-k-1} \times \# \star \sigma^{p-k}$ |
| discrete stress matrix | $\mathbb{T}$ | $\# \star \sigma^1 \times p$ |
| unit normal vectors | $\mathbb{N}$ | $p \times \#\sigma^1$ |
| body force matrix | $\mathbb{B}$ | $\#\sigma^1 \times \#\sigma^1$ (?) |
| acceleration matrix | $\mathbb{A}$ | $\#\sigma^1 \times \#\sigma^1$ (?) |

Balance of linear momentum reads:

$$\widetilde{\mathbb{M}}_2 \mathbb{T} + \mathbb{B} = \mathbb{A} \quad \text{(3D)}$$

$$\widetilde{\mathbb{M}}_1 \mathbb{T} + \mathbb{B} = \mathbb{A} \quad \text{(2D)}$$

Balance of angular momentum reads:

$$\mathbb{T}\mathbb{N} = 0$$

Compatibility equations can also be formulated.

"In this theory, the only metric-dependent matrices are $\mathbb{A}$ and $\mathbb{N}$; all the others are topological" [Yavari 2008]

# Outline

# Summary of Discrete Quantities

In this talk we have discussed only a few of the variables analyzed by the authors.

| Quantity | Symbol | Type |
|---|---|---|
| Velocity | $\vec{v}$ | vector-valued 0-form |
| Displacement | $\vec{u}$ | vector-valued 0-form |
| Strain | $\mathbb{F}$ | vector-valued 1-form |
| Mass desnity | $\rho$ | dual $p$-form |
| Internal energy density | $e$ | support volume-form |
| Specific entropy | N | support volume-form |
| Heat flux | $h$ | dual ($p$-1)-form |
| Heat supply | $r$ | dual $p$-form |
| Stress | $\mathcal{T}$ | covector-valued ($p$-1)-form |
| Body force | **b** | covector-valued dual $p$-form |
| Kinetic energy density | $\kappa$ | dual $p$-form |

*Notation from [Yavari 2008] for problems in dimension $p$ = 2 or 3.*

# Future Work in this field

- Implement this discrete algorithm.
- Compare implementation to existing finite element and other numerical methods for elasticity.
- Analyze convergence issues associated with this approach.

# References


A. Yavari *On geometric discretization of elasticity*, J. Mathematical Physics, 2008.

E. Kanso, M. Arroyo, Y. Tong, A. Yavari, J. Marsden, M. Desbrun *On the geometric character of stress in continuum mechanics*, Zeitschrift für angewandte Mathematik und Physik (ZAMP) 2007.

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A copy of these slides is available at

*http://www.math.utexas.edu/users/agillette/*