Geometric Criteria for Generalized Barycentric Finite Elements

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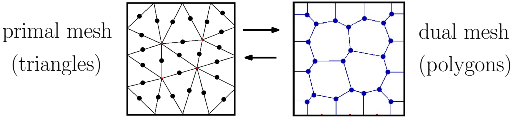
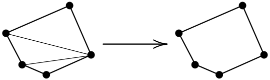
Outline

Why consider polygonal finite elements?

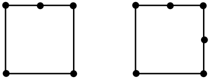
- **Theoretical:** Discrete Exterior Calculus considerations



  - Generic approach for problems with variables in duality.
- **Applied:** A new approach to longstanding meshing problems
  - Large angle removal by local remeshing



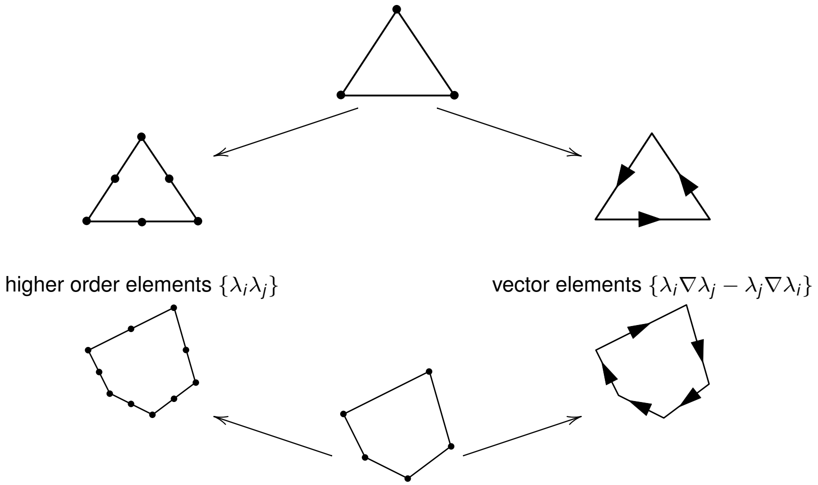
  - Canonical adaptive meshing elements



- **Practical:** General code would encompass old and new methods.

Overview of Approach

linear elements: $\{\lambda_i\}$ = (triangular) barycentric coordinates



linear elements: $\{\lambda_i\}$ = *generalized* barycentric coordinates

Definition

Let Ω be a convex polygon in $\mathbb{R}^2$ with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are called **barycentric coordinates** on Ω if they satisfy two properties:

1. **Non-negative**: $\lambda_i \geq 0$ on Ω.
2. **Linear Completeness**: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.

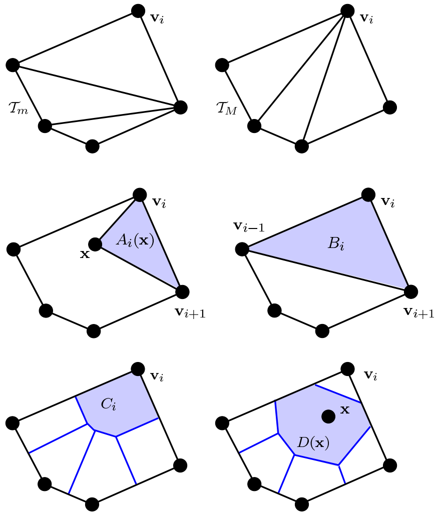
Any set of barycentric coordinates under this definition also satisfies:

3. **Partition of unity:** $\sum_{i=1}^{n} \lambda_i \equiv 1$.
4. **Linear precision:** $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.
5. **Interpolation:** $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

Theorem [Warren, 2003]

If the λ_i are rational functions of degree $n-2$, then they are unique.

Many generalizations to choose from . . .



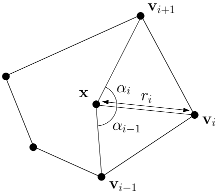
- Triangulation
  ⇒ FLOATER, HORMANN, KÓS, *A general construction of barycentric coordinates over convex polygons*, 2006

  $$0 \leq \lambda_i^{\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda_i^{\mathcal{T}_M}(\mathbf{x}) \leq 1$$

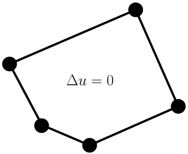
- Wachspress
  ⇒ WACHSPRESS, *A rational finite element basis*, 1975.

- Sibson
  ⇒ SIBSON, *A vector identity for the Dirichlet tessellation*, 1980.

Many generalizations to choose from . . .



- Mean value
  ⇒ FLOATER, *Mean value coordinates*, 2003.
  ⇒ FLOATER, KÓS, REIMERS, *Mean value coordinates in 3D*, 2005.



- Harmonic
  ⇒ WARREN, *Barycentric coordinates for convex polytopes*, 1996.
  ⇒ WARREN, SCHAEFER, HIRANI, DESBRUN, *Barycentric coordinates for convex sets*, 2007.

Many more in graphics contexts...

Outline

Optimal Convergence Estimates on Polygons

Let Ω be a convex polygon with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

For linear elements, an **optimal convergence estimate** has the form

$$\underbrace{\left|\left| u - \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C \operatorname{diam}(\Omega) \left|u\right|_{H^2(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^2(\Omega). \tag{1}$$

The **Bramble-Hilbert lemma** in this context says that any $u \in H^2(\Omega)$ is close to a first order polynomial in H^1 norm.

VERFÜRTH, *A note on polynomial approximation in Sobolev spaces*, Math. Mod. Num. An., 2008.
DEKEL, LEVIATAN, *The Bramble-Hilbert lemma for convex domains*, SIAM J. Math. An., 2004.

For (1), it suffices to prove an ***H^1*-interpolant estimate** over domains of diameter one:

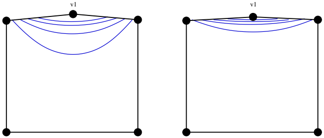
$$\left|\left| \sum_{i=1}^{n} u(\mathbf{v}_i)\lambda_i \right|\right|_{H^1(\Omega)} \leq C_I \left|\left|u\right|\right|_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \tag{2}$$

For (2), it suffices to **bound the gradients** of the $\{\lambda_i\}$, i.e. prove $\exists\, C_\lambda \in \mathbb{R}$ such that

$$\left|\left|\nabla \lambda_i\right|\right|_{L^2(\Omega)} \leq C_\lambda. \tag{3}$$

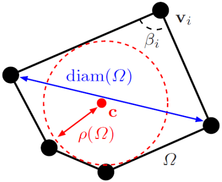
Geometric Hypotheses for Convergence Estimates

To bound the gradients of the coordinates, we need estimates of the geometry.



Let $\rho(\Omega)$ denote the radius of the largest inscribed circle. The **aspect ratio** γ is defined by

$$\gamma = \frac{\mathrm{diam}(\Omega)}{\rho(\Omega)} \in (2, \infty)$$



Three possible geometric conditions on a polygonal mesh:

G1. BOUNDED ASPECT RATIO: $\exists\ \gamma^* < \infty$ such that $\gamma < \gamma^*$

G2. MINIMUM EDGE LENGTH: $\exists\ d_* > 0$ such that $|\mathbf{v}_i - \mathbf{v}_{i-1}| > d_*$

G3. MAXIMUM INTERIOR ANGLE: $\exists\ \beta^* < \pi$ such that $\beta_i < \beta^*$

Summary of convergence results

Theorem

In the table below, any necessary geometric criteria to achieve the optimal convergence estimate are denoted by N. The set of geometric criteria denoted by S in each row are sufficient to guarantee estimate.

GILLETTE, RAND, BAJAJ *Error Estimates for Generalized Barycentric Interpolation*, Advances in Computational Mathematics, accepted, 2011.

		G1 aspect ratio	G2 min. edge	G3 max angle
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Harmonic	λ^{Har}	S	-	-
Future work:				
Mean Value	λ^{MV}	*S*	*S*	-

Implication of convergence results

restrictive — GEOMETRIC CONSTRAINTS — flexible

⟵⟶

Triangulate | Wachspress | Sibson | *Mean Value* | Harmonic

easy — EFFICIENT IMPLEMENTATION — difficult

⟵⟶

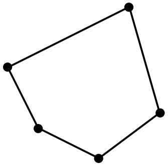
Triangulate | Wachspress | *Mean Value* | Sibson | Harmonic

Outline

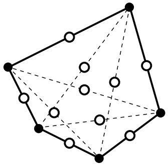
1. Motivation and Background
2. Linear Elements
3. Quadratic ‘Serendipity’ Elements
4. Vector Elements and Future Directions

From linear to quadratic elements

A naïve quadratic element is formed by products of linear element basis functions:



$$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\lambda_a\lambda_b\}$$



Why is this naïve?

- For an n-gon, this construction gives $n + \binom{n}{2}$ basis functions $\lambda_a\lambda_b$
- The space of quadratic polynomials is only dimension 6: $\{1, x, y, xy, x^2, y^2\}$
- Conforming to a linear function on the boundary requires 2 degrees of freedom per edge $\Rightarrow$ *only 2n functions needed!*

Problem Statement

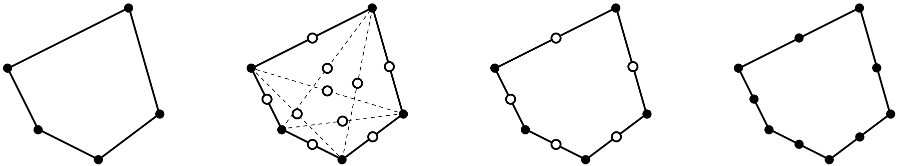
Construct $2n$ basis functions associated to the vertices and edge midpoints of an arbitrary n-gon such that a quadratic convergence estimate is obtained.

Polygonal Quadratic Serendipity Elements

We define matrices $\mathbb{A}$ and $\mathbb{B}$ to reduce the naïve quadratic basis.

filled dot $=$ Lagrangian domain point
$=$ all functions in the set evaluate to 0
except the associated function which evaluates to 1

open dot $=$ non-Lagrangian domain point
$=$ partition of unity satisfied, but not Lagrange property



$\{\lambda_i\} \xrightarrow[\text{products}]{\text{pairwise}} \{\mu_{ab}\} \xrightarrow{\mathbb{A}} \{\xi_{ij}\} \xrightarrow{\mathbb{B}} \{\psi_{ij}\}$

Linear	Quadratic	Serendipity	Lagrange
n	$n + \binom{n}{2}$	$2n$	$2n$

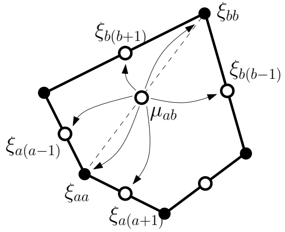
From quadratic to serendipity

Serendipity basis functions ξ_{ij} are constructed as a linear combination of pairwise product functions μ_{ab}:

$$[\xi_{ij}] = \mathbb{A}\begin{bmatrix} \mu_{aa} \\ \hline \mu_{a(a+1)} \\ \hline \mu_{ab} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & c_{ab}^{ij} \end{bmatrix} \begin{bmatrix} \mu_{aa} \\ \hline \mu_{a(a+1)} \\ \hline \mu_{ab} \end{bmatrix}$$

The quadratic basis is ordered as follows:

$$\begin{aligned} \mu_{aa} &= \text{basis functions associated with vertices} \\ \mu_{a(a+1)} &= \text{basis functions associated with edge midpoints} \\ \mu_{ab} &= \text{basis functions associated with interior diagonals,} \\ &\quad \text{i.e. } b \notin \{a-1, a, a+1\} \end{aligned}$$

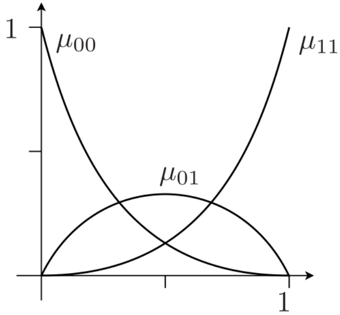


Theorem

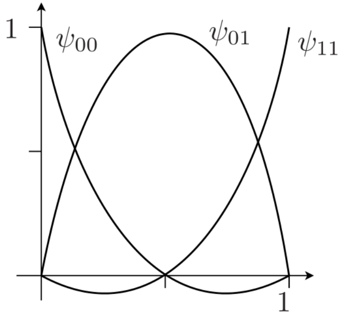
Constants $\{c_{ij}^{ab}\}$ exist for **any** convex polygon such that the resulting basis $\{\xi_{ij}\}$ satisfies constant, linear, and quadratic precision requirements.

Pairwise products vs. Lagrange basis

Pairwise products of barycentric functions do not form a Lagrange basis at interior degrees of freedom:



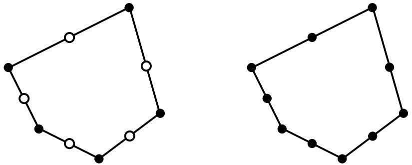
Pairwise products of barycentric functions



Lagrange basis

Translation between these two bases is straightforward and generalizes to the higher dimensional case...

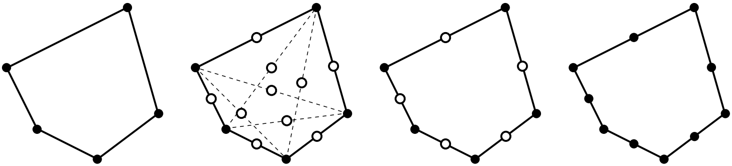
From serendipity to Lagrange



$$\{\xi_{ij}\} \xrightarrow{\quad\mathbb{B}\quad} \{\psi_{ij}\}$$

$$[\psi_{ij}] = \begin{bmatrix} \psi_{11} \\ \psi_{22} \\ \vdots \\ \psi_{nn} \\ \psi_{12} \\ \psi_{23} \\ \vdots \\ \psi_{1n} \end{bmatrix} = \left[\begin{array}{ccccc|ccccc} 1 & & & & & -1 & & \cdots & & -1 \\ & 1 & & & & -1 & -1 & \cdots & & \\ & & \ddots & & & & \ddots & \ddots & & \\ & & & \ddots & & & & \ddots & \ddots & \\ & & & & 1 & & & & -1 & -1 \\ \hline & & & & & 4 & & & & \\ & & & & & & 4 & & & \\ & & 0 & & & & & \ddots & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 4 \end{array}\right] \begin{bmatrix} \xi_{11} \\ \xi_{22} \\ \vdots \\ \xi_{nn} \\ \xi_{12} \\ \xi_{23} \\ \vdots \\ \xi_{1n} \end{bmatrix} = \mathbb{B}[\xi_{ij}].$$

Serendipity Theorem



Theorem

Given bounds on polygon aspect ratio (G1), minimum edge length (G2), and maximum interior angles (G3):

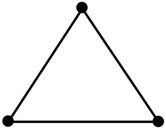
- $||\mathbb{A}||$ is uniformly bounded,
- $||\mathbb{B}||$ is uniformly bounded, and
- The basis $\{\psi_{ij}\}$ interpolates smooth data with $O(h^2)$ error.

RAND, GILLETTE, BAJAJ *Quadratic Serendipity Finite Element on Polygons Using Generalized Barycentric Coordinates*, Submitted, 2011

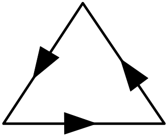
Outline

From scalar to vector elements

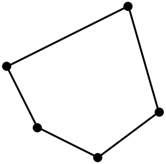
Barycentric functions are used to define $H(\text{curl})$ vector elements on triangles:



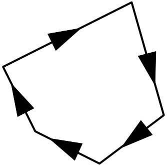
$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a\nabla\lambda_b - \lambda_b\nabla\lambda_a\}$$



Generalized barycentric functions provide $H(\text{curl})$ elements on polygons:



$$\{\lambda_i\} \xrightarrow[\text{construction}]{\text{Whitney}} \{\lambda_a\nabla\lambda_b - \lambda_b\nabla\lambda_a\}$$



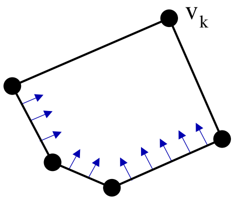
This idea fits naturally into the framework of **Discrete Exterior Calculus** and suggests a wide range of applications.

GILLETTE, BAJAJ *Dual Formulations of Mixed Finite Element Methods with Applications*
Computer-Aided Design 43:10, pages 1213-1221, 2011.

Conformity and interpolation properties

Conformity: The basis functions $\{\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i\}$ interpolate an $H(\text{curl})$ function.

Let $T_E\vec{v}$ denote the tangential projection of $\vec{v}$ to an edge E.



$$H(\text{curl}\,) := \left\{\vec{v} \in \left(L^2(\Omega)\right)^3 \quad \text{s.t.} \quad \nabla\times\vec{v} \in \left(L^2(\Omega)\right)^3\right\}$$

$$\vec{v} \in H(\text{curl}\,) \iff T_E\vec{v} \in C^0, \quad \forall \text{ edges } E \text{ in mesh}$$

$$\lambda_k \equiv 0 \quad \text{on } E \not\ni v_k$$

$$\therefore \nabla\lambda_k \perp E \quad \text{on } E \not\ni v_k$$

$$\therefore \ T_E(\lambda_i\nabla\lambda_j) \neq 0 \iff \mathbf{v}_i, \mathbf{v}_j \in E$$

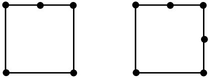
Interpolation: The basis functions are Lagrange-like for edge integrals.

$$T_{\vec{e_{ij}}}(\nabla\lambda_i) = \frac{1}{|e_{ij}|}, \quad \text{since the } \lambda_i \text{ are linear on edges.}$$

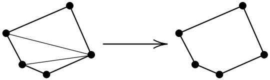
$$\int_{e_{ij}} (\lambda_i\nabla\lambda_j - \lambda_j\nabla\lambda_i)\cdot\vec{e_{ij}} = \frac{1}{|e_{ij}|}\int_{e_{ij}} \lambda_i + \lambda_j = \frac{1}{|e_{ij}|}\int_{e_{ij}} 1 = 1.$$

Future work and open problems

- **Theoretical:**
  - Higher order elements (degree $p > 2$).
  - Spline / iso-geometric theory for generic polygonal elements.
  - Extension to 3D generalized barycentric functions.
- **Applied:**
  - Employ degenerate polygons as adaptive elements:



  - Apply gradient bounds to aid with image processing / computer graphics problems
- **Practical:**
  - Create finite element codes which automatically replace large angle triangles with generalized barycentric polygonal elements.



Questions?

- Slides and pre-prints available at http://ccom.ucsd.edu/∼agillette
- Thanks for the invitation to speak!