Nodal Basis Functions for Serendipity Finite Elements

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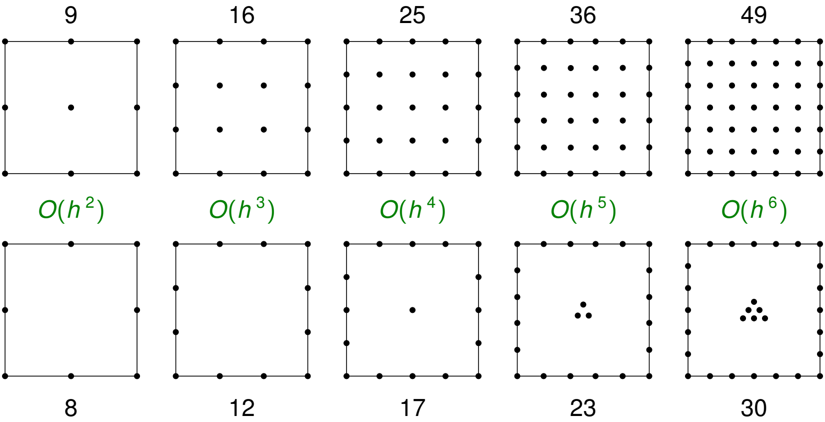
joint work with Michael Floater (U. Oslo)

Serendipity finite elements provide a convenient means to reduce the computational effort required for higher order tensor-product methods. The number of basis functions supported on a tensor-product element of order r in n dimensions is $(r+1)^n$, while the corresponding serendipity element has asymptotically $\sim r^n/n!$ for large r, representing a reduction of 50% in 2-D and 83% in 3-D. We construct basis functions for serendipity elements of any order $r \geq 1$ in any number of dimensions $n \geq 1$ that are interpolatory at user-specified nodes and can be written as linear combinations of standard tensor-product polynomials.

Tensor-product vs. serendipity degrees of freedom

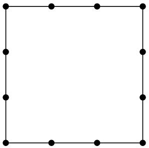


Elements in the same column exhibit the same rate of convergence:

$$\underbrace{||u - u_h||_{H^1(\Omega)}}_{\text{approximation error}} \leq \underbrace{C\, h^{\mathbf{r}}\, |u|_{H^{\mathbf{r+1}}(\Omega)}}_{\text{optimal error bound}}, \quad \forall u \in H^{\mathbf{r+1}}(\Omega).$$

This ‘serendipitous’ observation led to the namesake of the elements.

Characterization of requisite monomials



total degree at most 3 (dim=10)

$$\{1, x, y, x^2, y^2, xy, x^3, y^3, x^2y, xy^2, x^3y, xy^3, x^2y^2, x^3y^2, x^2y^3, x^3y^3\}$$

superlinear degree at most 3 (dim=12)

at most degree 3 in each variable (dim=16)

The **superlinear** degree of a polynomial ignores linearly-appearing variables.

Example: $\mathrm{sldeg}(xy^3) = 3$, even though $\deg(xy^3) = 4$

Definition: $\mathrm{sldeg}(x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}) := \left(\sum_{i=1}^{n} e_i \right) - \#\{e_i \ : \ e_i = 1\}$

Observe that the set $S_r = \left\{\alpha \in \mathbb{N}_0^n : \mathrm{sldeg}(x^\alpha) \le r\right\}$ has a partial ordering

and is thus a **lower set**, meaning $\boxed{\alpha \in S_r,\ \mu \le \alpha \implies \mu \in S_r}$

ARNOLD, AWANOU *The serendipity family of finite elements*, FoCM, 2011.

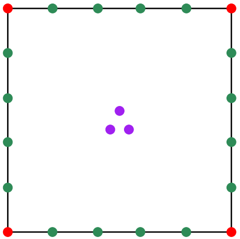
Partitioning and reordering the multi-indices

Theorem

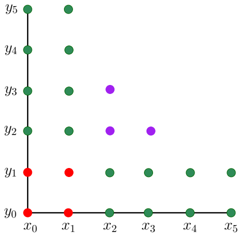
Fix a lower set $L \subset \mathbb{N}_0^n$ and points $z_\alpha \in \mathbb{R}^n$ for all $\alpha \in L$. For any sufficiently smooth n-variate real function f, there is a *unique polynomial p* in $\text{span}\{x^\alpha : \alpha \in L\}$ that *interpolates f at the points z_α*, with partial derivative interpolation for repeated z_α values.

DYN, FLOATER, *Multivariate polynomial interpolation on lower sets*, J. Approx. Th., to appear.

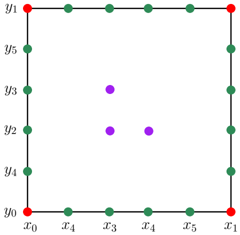
We apply the above theorem in the context of the lower set S_r:



The order 5 serendipity element, with degrees of freedom color-coded by dimensionality.



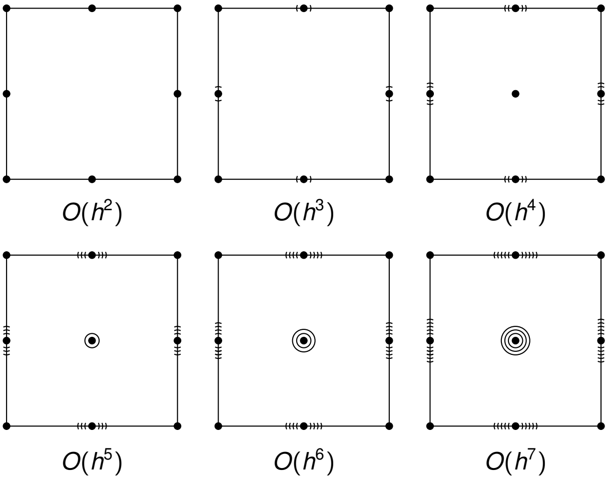
The lower set S_5, with equivalent color coding.



The lower set S_5, with domain points z_α reordered.

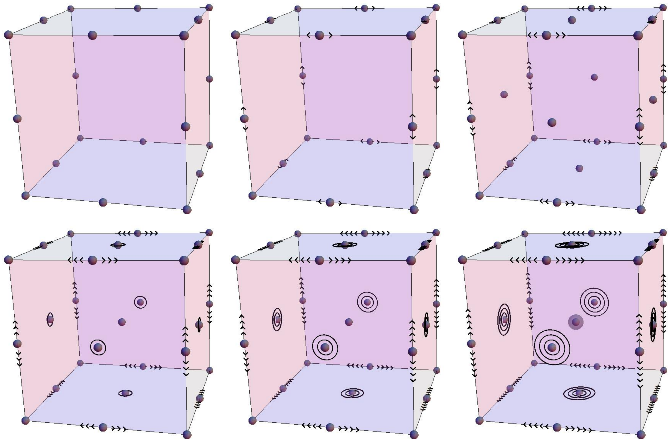
2D symmetric serendipity elements

We generate symmetric $O(h^r)$ serendipity elements on $[-1,1]^2$ by setting $x_j = y_j = 0$ for $2 \leq j \leq r$. This approach interpolates partial derivative information at the edge midpoints and square center.



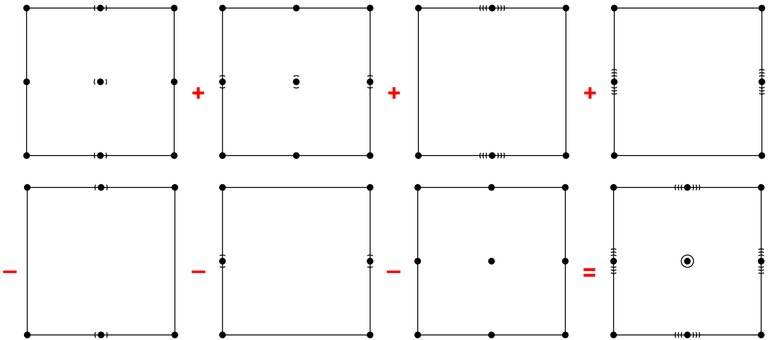
3D symmetric serendipity elements

The approach applies without modification to any dimension $n \geq 1$.



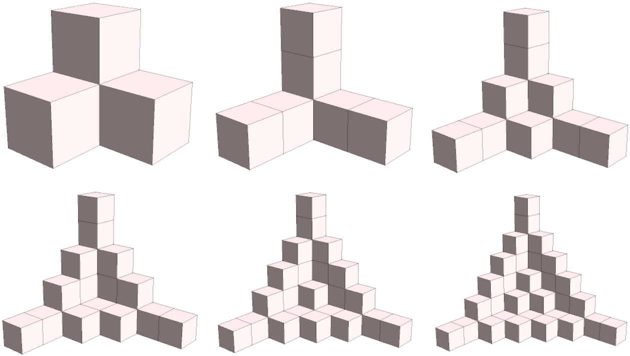
Linear combinations of tensor-products

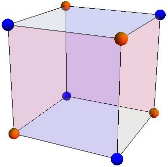
The Dyn-Floater theorem provides an explicit formula for computing the desired nodal interpolation scheme as a linear combination of standard tensor-product functions.



Computing coefficients of linear combinations

Lower sets corresponding to S_2 through S_7 in 3 variables.





3D coefficient calculator

The linear combination is the sum over all blocks within the lower set with coefficients determined as follows:

$\rightarrow$ Place the coefficient calculator at an extremal point of S_r.

$\rightarrow$ Add up the values for all vertices appearing in S_r.

(blue $\rightarrow$ +1;   orange $\rightarrow$ -1).

$\rightarrow$ The coefficient for the block is the value of the sum.

Citations and Acknowledgements

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More information: `http://math.arizona.edu/~agillette/`

N.B. The ICERM *logo is a lower set in 3D!*