Error Estimates for Generalized Barycentric Interpolation

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1. Abstract

We prove the optimal convergence estimate for first order interpolants used in finite element methods based on three major approaches for generalizing barycentric interpolation functions to convex planar polygonal domains. The Wachspress approach explicitly constructs rational functions, the Sibson approach uses Voronoi diagrams on the vertices of the polygon to define the functions, and the Harmonic approach defines the functions as the solution of a PDE. We show that given certain conditions on the geometry of the polygon, each of these constructions can obtain the optimal convergence estimate. In particular, we show that the well-known maximum interior angle condition required for interpolants over triangles is still required for Wachspress functions but not for Sibson functions.

2. Interpolation in Sobolev Spaces

A set of barycentric coordinates $\{\lambda_i\}$ for a polygonal domain Ω is associated with the interpolation operator $I : H^2(\Omega) \to \text{span}\{\lambda_i\} \subset H^1(\Omega)$ given by

$$Iu := \sum_i u(\mathbf{v}_i)\lambda_i. \qquad (1)$$

The **optimal convergence estimate** for a finite element method using this interpolant is

$$||u - Iu||_{H^1(\Omega)} \leq C\, \text{diam}(\Omega)\, |u|_{H^2(\Omega)}, \quad \forall u \in H^2(\Omega). \qquad (2)$$

We restrict to the case where Ω is a convex domain with diameter 1. By the Bramble-Hilbert Lemma, it suffices to prove the H^1 **interpolant estimate**

$$||Iu||_{H^1(\Omega)} \leq C_I\, ||u||_{H^2(\Omega)}, \quad \forall u \in H^1(\Omega). \qquad (3)$$

Our approach is to identify necessary and sufficient geometric restrictions on Ω to guarantee that (2) holds for various generalizations of barycentric functions.

3. Generalized Barycentric Coordinate Types

Definition 1. *Functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ are* ***barycentric coordinates*** *on Ω if they satisfy two properties.*

B1. ***Non-negative****: $\lambda_i \geq 0$ on Ω.*

B2. ***Linear Completeness****: For any linear function $L : \Omega \to \mathbb{R}$, $L = \sum_{i=1}^{n} L(\mathbf{v}_i)\lambda_i$.*

The following properties can be shown to hold for any functions satisfying Definition 1:

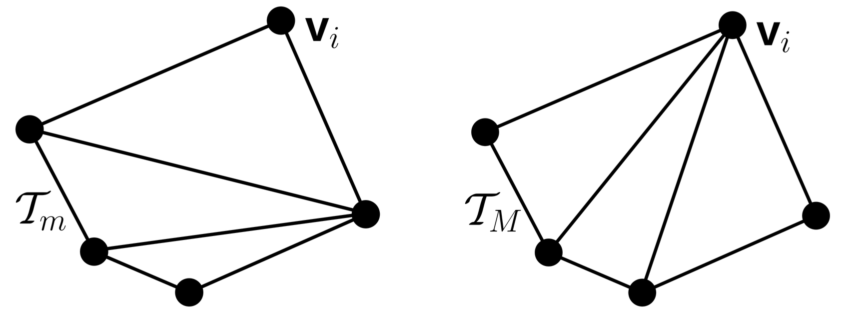
B3. **Partition of unity**: $\sum_{i=1}^{n} \lambda_i \equiv 1$.

B4. **Linear precision**: $\sum_{i=1}^{n} \mathbf{v}_i \lambda_i(\mathbf{x}) = \mathbf{x}$.

B5. **Interpolation**: $\lambda_i(\mathbf{v}_j) = \delta_{ij}$.

We restrict to coordinates which are invariant under rotation, translation, and uniform scaling.

Triangulation Coordinates



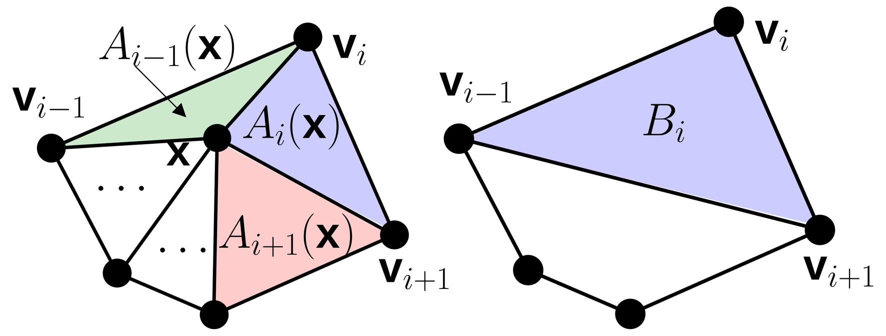
Let $\mathcal{T}$ be a triangulation of Ω.

$\lambda^{\text{Tri}}_{i,\mathcal{T}}(\mathbf{x}) :=$ standard barycentric function associated to $\mathbf{v}_i$ on $\mathcal{T}$

Proposition: (Floater et al. [2]) Any barycentric coordinate function λ_i satisfies the bounds

$$0 \leq \lambda^{\text{Tri}}_{i,\mathcal{T}_m}(\mathbf{x}) \leq \lambda_i(\mathbf{x}) \leq \lambda^{\text{Tri}}_{i,\mathcal{T}_M}(\mathbf{x}) \leq 1, \quad \forall \mathbf{x} \in \Omega.$$

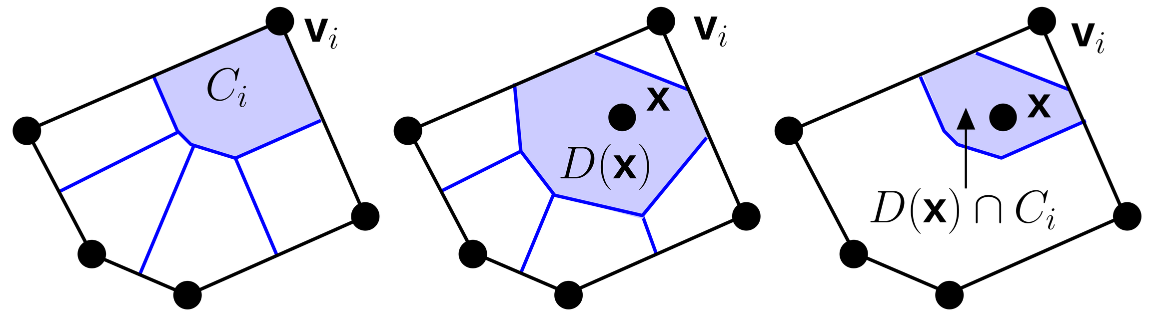
Wachspress Coordinates



$$w_i^{\text{Wach}}(\mathbf{x}) := B_i \prod_{j \neq i, i-1} A_j(\mathbf{x}).$$

$$\lambda_i^{\text{Wach}}(\mathbf{x}) := \frac{w_i^{\text{Wach}}(\mathbf{x})}{\sum_{j=1}^{n} w_j^{\text{Wach}}(\mathbf{x})}$$

Sibson (Natural Neighbor) Coordinates



$$P := \{\mathbf{v}_i\}$$
$$P' := P \cup \{\mathbf{x}\}$$

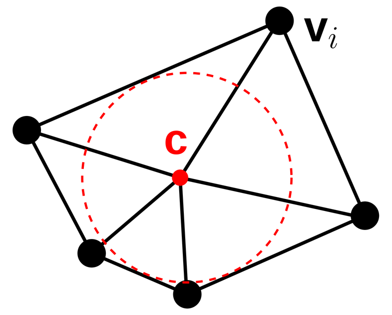
$$C_i := |V_P(\mathbf{v}_i) \cap \Omega|$$
$$D(\mathbf{x}) := |V_{P'}(\mathbf{x}) \cap \Omega|$$

$$\lambda_i^{\text{Sibs}}(\mathbf{x}) := \frac{D(\mathbf{x}) \cap C_i}{D(\mathbf{x})}$$

or, equivalently,

$$\lambda_i^{\text{Sibs}}(\mathbf{x}) = \frac{D(\mathbf{x}) \cap C_i}{\sum_{j=1}^{n} D_j(\mathbf{x}) \cap C_j}.$$

Harmonic Coordinates



Let $g_i : \partial\Omega \to \mathbb{R}$ be the piecewise linear function satisfying

$$g_i(\mathbf{v}_j) = \delta_{ij}, \quad g_i \text{ linear on each edge of } \Omega.$$

Define λ_i^{Har} as the solution to $\begin{cases} \Delta\left(\lambda_i^{\text{Har}}\right) = 0, & \text{on } \Omega, \\ \lambda_i^{\text{Har}} = g_i, & \text{on } \partial\Omega. \end{cases}$

Equivalently, $\lambda_i^{\text{Har}} := \text{argmin}\left\{ |\lambda|_{H^1(\Omega)} \;:\; \lambda = g_i \text{ on } \partial\Omega \right\}.$

4. Results

We consider three typical geometric criteria on Ω:

$\rho(\Omega) :=$ radius of the largest inscribed circle in Ω

$\beta_i :=$ interior angle at vertex $\mathbf{v}_i$.

G1. **Bounded aspect ratio:** There exists $\gamma^* \in \mathbb{R}$ such that $\frac{\text{diam}(\Omega)}{\rho(\Omega)} < \gamma^*$.

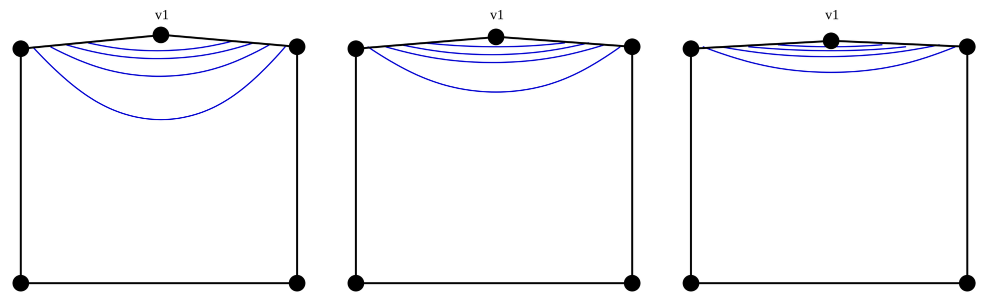
G2. **Minimum edge length:** There exists $d_* \in \mathbb{R}$ such that $|\mathbf{v}_i - \mathbf{v}_j| > d_* > 0$ for all $i \neq j$.

G3. **Maximum interior angle:** There exists $\beta^* \in \mathbb{R}$ such that $\beta_i < \beta^* < \pi$ for all i.

Theorem: In the table below, any necessary geometric criteria to achieve the optimal interpolant estimate (2) are denoted by N. The set of geometric criteria denoted by S in each row are sufficient to guarantee estimate (3).

		G1	G2	G3
Triangulated	λ^{Tri}	-	-	S,N
Wachspress	λ^{Wach}	S	S	S,N
Sibson	λ^{Sibs}	S	S	-
Harmonic	λ^{Har}	S	-	-

Example showing necessity of G3 for obtaining (2) with λ_i^{Wach}. Level sets of λ_1^{Wach} are shown.



5. Discussion and Future Work

The table provides a guideline for how to choose barycentric basis functions given geometric criteria or, conversely, which geometric criteria should be guaranteed given a choice of basis functions. Further investigations include analyzing the optimal error estimate for other generalizations of barycentric coordinates such as mean value and discrete harmonic coordinates, as well as extending the results to vector interpolants on polygons and interpolants on polytopes in $\mathbb{R}^3$.

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