An Introduction to Trimmed Serendipity Finite Element Spaces
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joint work with
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Outline

1. Four well-known families of finite elements
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Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

\[ n \rightarrow \text{the spatial dimension of the domain} \]
\[ r \rightarrow \text{the order of error decay} \]
\[ k \rightarrow \text{the differential form order of the solution space} \]

An element type is defined in part by its degrees of freedom. Typically:

*the more degrees of freedom, the greater the computational cost of the method*

**Ex:** \( Q_1^- \Lambda^2(\Box_3) \) is an element for

\[ n = 3 \quad \rightarrow \quad \text{domains in } \mathbb{R}^3 \]
\[ r = 1 \quad \rightarrow \quad \text{linear order of error decay} \]
\[ k = 2 \quad \rightarrow \quad \text{conformity in } \Lambda^2(\mathbb{R}^3) \sim H(\text{div}) \]

\( Q_1^- \Lambda^2(\Box_3) \) is part of the \( Q^- \) ‘column’ of elements,

is defined on geometry \( \Box_3 \) (i.e. a cube),

has a 6 dimensional space of test functions,

and has an associated set of 6 degrees of freedom

that are unisolvent for the test function space.
The ‘Periodic Table of the Finite Elements’


Classification of many common conforming finite element types.

\( n \rightarrow \) Domains in \( \mathbb{R}^2 \) (top half) and in \( \mathbb{R}^3 \) (bottom half)

\( r \rightarrow \) Order 1, 2, 3 of error decay (going down columns)

\( k \rightarrow \) Conformity type \( k = 0, \ldots, n \) (going across a row)

Geometry types: Simplices (left half) and cubes (right half).
Raviart, Thomas, “A mixed finite element method for 2nd order elliptic problems” Lecture Notes in Mathematics, 1977 ← 3172 citations, including 150 from 2017!

Nédélec, “Mixed finite elements in \( \mathbb{R}^3 \),” Numerische Mathematik, 1980


Nédélec, “A new family of mixed finite elements in \( \mathbb{R}^3 \),” Numerische Mathematik, 1986


Arnold, Awano “Finite element differential forms on cubical meshes”, Math Comp., 2013

Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

Unstable method

\[ \begin{align*}
\mathbf{P}_1 \subset \mathcal{H}^1 \times \mathcal{H}^1 & \subset \mathcal{L}^2 \\
2 & \Rightarrow \\
\text{Unstable method}
\end{align*} \]

Provably stable method

\[ \begin{align*}
\mathbf{RT}_{1}^{0/1} \subset \mathcal{H}(\text{div}) \subset \mathcal{L}^2 \\
3 & \Rightarrow \\
\text{Provably stable method} & \text{converges to} \\
u = x(1 - x)y(1 - y)
\end{align*} \]

Example and images on right from:

Provably stable method converges to 
\[ u = x(1 - x)y(1 - y) \]

Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the \( L^2 \) deRham Diagram.

\[
\begin{array}{c}
H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \\
\text{vector Poisson} & \sigma & \mu \\
\text{Maxwell’s eqn’s} & h & b \\
\text{Darcy / Poisson} & u & p
\end{array}
\]

The Periodic Table of Finite Elements lets us ‘read off’ stable pairs visually.
Stable pairs for tetrahedral meshes

Problem: Darcy / Poisson
Dimension: $n = 3$
Mesh type: tetrahedral
Convergence: quadratic ($r = 2$)
Stable pairs for tetrahedral meshes

**Problem:** Darcy / Poisson

**Dimension:** $n = 3$

**Mesh type:** tetrahedral

**Convergence:** cubic ($r = 3$)
Stable pairs for tetrahedral meshes

Problem: Maxwell’s
Dimension: \( n = 3 \)
Mesh type: tetrahedral
Convergence: quadratic (\( r = 2 \))
Stable pairs for tetrahedral meshes

Problem: Maxwell’s
Dimension: $n = 3$
Mesh type: tetrahedral
Convergence: cubic ($r = 3$)
Stable pairs for tetrahedral meshes

**Problem:** vector Poisson

**Dimension:** $n = 3$

**Mesh type:** tetrahedral

**Convergence:** quadratic ($r = 2$)
Problem: vector Poisson
Dimension: $n = 3$
Mesh type: tetrahedral
Convergence: cubic ($r = 3$)
Stable pairs for cubical meshes

Problem: Darcy / Poisson
Dimension: $n = 3$
Mesh type: cubes
Convergence: quadratic ($r = 2$)
Problem: Darcy / Poisson
Dimension: $n = 3$
Mesh type: cubes
Convergence: cubic ($r = 3$)
Problem: Maxwell’s
Dimension: $n = 3$
Mesh type: cubes
Convergence: quadratic ($r = 2$)
Stable pairs for cubical meshes

Problem: Maxwell’s
Dimension: $n = 3$
Mesh type: cubes
Convergence: cubic ($r = 3$)
Problem: vector Poisson
Dimension: $n = 3$
Mesh type: cubes
Convergence: quadratic ($r = 2$)
Problem: vector Poisson
Dimension: \( n = 3 \)
Mesh type: cubes
Convergence: cubic (\( r = 3 \))
Exact cochain complexes found in the table

On an $n$-simplex in $\mathbb{R}^n$:

- Trimmed polynomials:
  \[ \mathcal{P}_r^{-} \Lambda^0 \to \mathcal{P}_r^{-} \Lambda^1 \to \cdots \to \mathcal{P}_r^{-} \Lambda^{n-1} \to \mathcal{P}_r^{-} \Lambda^n \]

- Polynomials:
  \[ \mathcal{P}_r \Lambda^0 \to \mathcal{P}_{r-1} \Lambda^1 \to \cdots \to \mathcal{P}_{r-n+1} \Lambda^{n-1} \to \mathcal{P}_{r-n} \Lambda^n \]

On an $n$-dimensional cube in $\mathbb{R}^n$:

- Tensor product:
  \[ \mathcal{Q}_r^{-} \Lambda^0 \to \mathcal{Q}_r^{-} \Lambda^1 \to \cdots \to \mathcal{Q}_r^{-} \Lambda^{n-1} \to \mathcal{Q}_r^{-} \Lambda^n \]

- Serendipity:
  \[ \mathcal{S}_r \Lambda^0 \to \mathcal{S}_{r-1} \Lambda^1 \to \cdots \to \mathcal{S}_{r-n+1} \Lambda^{n-1} \to \mathcal{S}_{r-n} \Lambda^n \]

The ‘minus’ spaces proceed across rows of the PToFE ($r$ is fixed) while the ‘regular’ spaces proceed along diagonals ($r$ decreases).

Mysteriously, the degree of freedom count for mixed methods from the $\mathcal{P}_r^-$ spaces is smaller than those from the $\mathcal{P}_r$ spaces, while the opposite is true for the $\mathcal{Q}_r^-$ and $\mathcal{S}_r$ spaces.
Outline

1. Four well-known families of finite elements
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3. A computational basis for trimmed serendipity spaces
The 5th column: Trimmed serendipity spaces

A new column for the PToFE: the **trimmed serendipity** elements.

\[ S_r^{-} \Lambda^k(\square_n) \]

denotes approximation order \( r \),
subset of \( k \)-form space \( \Lambda^k(\Omega) \),
use on meshes of \( n \)-dim'l cubes.

Defined for any \( n \geq 1, 0 \leq k \leq n, r \geq 1 \)

Identical or analogous properties to all the other columns in the table.

The advantage of the \( S_r^{-} \Lambda^k \) spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.
Key properties of the trimmed serendipity spaces

\[ Q_r^{-} \Lambda^0 \rightarrow Q_r^{-} \Lambda^1 \rightarrow \cdots \rightarrow Q_r^{-} \Lambda^{n-1} \rightarrow Q_r^{-} \Lambda^n \]  \hspace{1cm} \text{tensor product} \\
\[ S_r \Lambda^0 \rightarrow S_{r-1} \Lambda^1 \rightarrow \cdots \rightarrow S_{r-n+1} \Lambda^{n-1} \rightarrow S_{r-n} \Lambda^n \]  \hspace{1cm} \text{serendipity} \\
\[ S_r^{-} \Lambda^0 \rightarrow S_r^{-} \Lambda^1 \rightarrow \cdots \rightarrow S_r^{-} \Lambda^{n-1} \rightarrow S_r^{-} \Lambda^n \]  \hspace{1cm} \text{trimmed serendipity}

\textbf{Subcomplex:} \hspace{0.5cm} dS_r^{-} \Lambda^k \subset S_r^{-} \Lambda^{k+1}

\textbf{Exactness:} \hspace{0.5cm} The above sequence is \textit{exact}. \hspace{0.5cm} i.e. the image of incoming map = kernel of outgoing map

\textbf{Inclusion:} \hspace{0.5cm} S_r \Lambda^k \subset S_{r+1}^{-} \Lambda^k \subset S_{r+1} \Lambda^k

\textbf{Trace:} \hspace{0.5cm} \text{tr}_f S_r^{-} \Lambda^k (\mathbb{R}^n) \subset S_r^{-} \Lambda^k (f), \hspace{0.5cm} \text{for any} \hspace{0.5cm} (n-1)-\text{hyperplane} \hspace{0.5cm} f \hspace{0.5cm} \text{in} \hspace{0.5cm} \mathbb{R}^n

\textbf{Special cases:} \hspace{0.5cm} S_r^{-} \Lambda^0 = S_r \Lambda^0 \hspace{0.5cm} S_r^{-} \Lambda^n = S_{r-1} \Lambda^n \hspace{0.5cm} S_r^{-} \Lambda^k + dS_{r+1} \Lambda^{k-1} = S_r \Lambda^k.

Replace ‘S’ by ‘P’ \( \rightarrow \) key properties about the first two columns for \( P_r^{-} \Lambda^k \) and \( P_r \Lambda^k \)!
Dimension count and comparison

Formula for counting degrees of freedom of $S_r^- \wedge^k (\Box_n)$:

$$\min\{n, \lfloor r/2 \rfloor + k\} \sum_{d=k}^{2n-d} \binom{n}{d} \left( \binom{r-d+2k-1}{r-d+k-1} \binom{d-k}{k} + \binom{r-d+2k}{d-k-1} \right)$$

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Mixed Method dimension comparison 1

Mixed method for Darcy problem:

\[ \mathbf{u} + K \nabla p = 0 \]
\[ \text{div } \mathbf{u} - f = 0 \]

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order \( r \) decay in the approximation of \( p, \mathbf{u}, \text{ and div } \mathbf{u} \) is desired.

### Total # of degrees of freedom on a square (\( n = 2 \)):

| \( r \) | \( |Q_r^{-} \Lambda^1| + |Q_r^{-} \Lambda^2| \) | \( |S_r \Lambda^1| + |S_{r-1} \Lambda^2| \) | \( |S_r^{-} \Lambda^1| + |S_r^{-} \Lambda^2| \) |
|---|---|---|---|
| 1 | 4+1 = 5 | 8+1 = 9 | 4+1 = 5 |
| 2 | 12+4 = 16 | 14+3 = 17 | 10+3 = 13 |
| 3 | 24+9 = 33 | 22+6 = 28 | 17+6 = 23 |

### Total # of degrees of freedom on a cube (\( n = 3 \)):

| \( r \) | \( |Q_r^{-} \Lambda^2| + |Q_r^{-} \Lambda^3| \) | \( |S_r \Lambda^2| + |S_{r-1} \Lambda^3| \) | \( |S_r^{-} \Lambda^2| + |S_r^{-} \Lambda^3| \) |
|---|---|---|---|
| 1 | 6+1 = 7 | 18+1 = 19 | 6+1 = 7 |
| 2 | 36+8 = 44 | 39+4 = 43 | 21+4 = 25 |
| 3 | 108+27 = 135 | 72+10 = 82 | 45+10 = 55 |
Mixed method for Darcy problem: \[ u + K \nabla p = 0 \]
\[ \text{div } u - f = 0 \]

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to trimmed serendipity:

### # of interior degrees of freedom on a square \((n = 2)\):

| \( r \) | \( |Q_r^{-1}\Lambda_0^1| + |Q_r^{-2}\Lambda_0^2| \) | \( |S_r\Lambda_0^1| + |S_{r-1}\Lambda_0^2| \) | \( |S_r^{-1}\Lambda_0^1| + |S_r^{-2}\Lambda_0^2| \) |
|-------|---------------------------------|------------------|------------------|
| 1     | 0+1 = 1                         | 0+1 = 1          | 0+1 = 1          |
| 2     | 4+4 = 8                         | 2+3 = 5          | 2+3 = 5          |
| 3     | 12+9 = 21                       | 6+6 = 12         | 5+6 = 11         |

### # of interior degrees of freedom on a cube \((n = 3)\):

| \( r \) | \( |Q_r^{-2}\Lambda_0^2| + |Q_r^{-3}\Lambda_0^3| \) | \( |S_r\Lambda_0^2| + |S_{r-1}\Lambda_0^3| \) | \( |S_r^{-2}\Lambda_0^2| + |S_r^{-3}\Lambda_0^3| \) |
|-------|---------------------------------|------------------|------------------|
| 1     | 0+1 = 1                         | 0+1 = 1          | 0+1 = 1          |
| 2     | 12+8 = 20                       | 3+4 = 7          | 3+4 = 7          |
| 3     | 54+27 = 81                      | 12+10 = 22       | 9+10 = 19        |
Mixed method for Darcy problem:
\[
\begin{align*}
\mathbf{u} + K \nabla p &= 0 \\
\text{div } \mathbf{u} - f &= 0
\end{align*}
\]

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements still have the fewest DoFs:

# of **interface** (edge) degrees of freedom on a square ($n = 2$):

| $r$ | $|Q_r \Lambda^1(\partial \square_2)|$ | $|S_r \Lambda^1(\partial \square_2)|$ | $|S_r^- \Lambda^1(\partial \square_2)|$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| 1   | 4                                | 8                                | 4                                |
| 2   | 8                                | 12                               | 8                                |
| 3   | 12                               | 16                               | 12                               |

# of **interface** (edge+face) degrees of freedom on a cube ($n = 3$):

| $r$ | $|Q_r \Lambda^2(\partial \square_3)|$ | $|S_r \Lambda^2(\partial \square_3)|$ | $|S_r^- \Lambda^2(\partial \square_3)|$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| 1   | 6                                | 18                               | 6                                |
| 2   | 24                               | 36                               | 18                               |
| 3   | 54                               | 60                               | 36                               |
1. Four well-known families of finite elements

2. Trimmed serendipity spaces: a new, smaller family of elements

3. A computational basis for trimmed serendipity spaces
Decomposition by polynomial subspace

$S_r^{-} \Lambda^k(\square_n)$ is a space of differential $k$-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$S_r^{-} \Lambda^k = P_{r}^{-} \Lambda^k \oplus J_r \Lambda^k \oplus dJ_{r} \Lambda^{k-1}$$

Polynomial coefficients in each summand:

- $P_{r}^{-} \Lambda^k$: anything up to degree $r - 1$ and some degree $r$
- $J_r \Lambda^k$: certain polynomials whose degree is between $r+1$ and $r+n-k-1$
- $dJ_{r} \Lambda^{k-1}$: certain polynomials whose degree is between $r$ and $r+n-k-2$

The “regular” serendipity space has an analogous decomposition:

$$S_r \Lambda^k = P_{r} \Lambda^k \oplus J_r \Lambda^k \oplus dJ_{r+1} \Lambda^{k-1}$$

This decomposition provides a direct sum into some precise but elaborate subspaces:

$$J_r \Lambda^k(\mathbb{R}^n) := \sum_{l \geq 1} \kappa \mathcal{H}_{r+l-1,l} \Lambda^{k+1}(\mathbb{R}^n),$$

where $\mathcal{H}_{r,l} \Lambda^k(\mathbb{R}^n) := \{ \omega \in \mathcal{H}_{r} \Lambda^k(\mathbb{R}^n) | \text{ldeg } \omega \geq l \}$,

where $\text{ldeg}(x^\alpha dx_\sigma) := \#\{i \in \sigma^* : \alpha_i = 1\}$.
We can also decompose $S_r^{-} \Lambda^k(\square_n)$ by the subspace of “zero trace”:

$$S_r^{-} \Lambda^k = S_r^{-} \Lambda^k_0 \oplus \left(S_r^{-} \Lambda^k_0\right)^\perp$$

We use this decomposition to prove that $S_r^{-} \Lambda^\bullet(\square_n)$ is a **minimal compatible finite element system** containing $\mathcal{P}_{r-1} \Lambda^\bullet(\square_n)$.

---

A computational basis for $S_r^{-} \Lambda^k_0$ would aid in the construction of bases for $S_r^{-} \Lambda^k$.

Building such a basis is non-trivial. Consider

$$\alpha := (z - 1)(y^2 - 1) \, dx + y(x + 1)(z - 1) \, dy + (x + 1)(y^2 - 1) \, dz$$

$$\beta := (z - 1)(y^2 - 1) \, dx - 2y(x + 1)(z - 1) \, dy + (x + 1)(y^2 - 1) \, dz$$

Both $\alpha$ and $\beta$ have a natural association to the approximation of $y$ on the edge $\{x = 1, z = -1\}$, and both are elements of $Q_2 \Lambda^1(\square_3)$. But *only* $\beta$ is in $S_1 \Lambda^1(\square_3)$!
Decompositions shared insight

Why is $\beta \in S_1 \Lambda^1(\square_3)$?

$$\beta = (z - 1)(y^2 - 1) \, dx$$
$$\quad - 2y(x + 1)(z - 1) \, dy$$
$$\quad + (x + 1)(y^2 - 1) \, dz$$

$$= y^2 z \, dx \quad -y^2 \, dx \quad 0 \, dx \quad (-z + 1) \, dx$$
$$-2xyz \, dy \quad +2xy \, dy \quad -2yz \, dy \quad 2y \, dy$$
$$xy^2 \, dz \quad 0 \, dz \quad y^2 \, dz \quad (-x - 1) \, dz$$

basis elements for $dJ_2 \Lambda^0$

$\in \mathcal{P}_1 \Lambda^1$

From the polynomial subspace decompositions:

$\beta \in dJ_2 \Lambda^0 \oplus \mathcal{P}_1 \Lambda^1 \subset S_1 \Lambda^1$

$\beta \in dJ_2 \Lambda^0 \oplus \mathcal{P}_1 \Lambda^1 \subset S_2^- \Lambda^1$

Full report on this approach coming soon!
Acknowledgments

Related Publication


Related talk

Victoria Sanders, University of Arizona
*Using Sage to Create Lists of Shape Functions for Trimmed Serendipity and Serendipity Finite Elements*
**Saturday 3:00 p.m. Room 19**

Research Funding

Supported in part by the National Science Foundation grant DMS-1522289.

Slides and Pre-prints

http://math.arizona.edu/~agillette/

Thanks to the organizers for the invitation to speak!