(Trimmed) Serendipity Finite Element Methods in Theory and Practice
Andrew Gillette - University of Arizona

joint work with
Tyler Kloefkorn, AAAS STP Fellow, hosted at NSF
Victoria Sanders, University of Arizona
1. The “Periodic Table of the Finite Elements”
2. How to find new finite elements by counting
3. Trimmed serendipity finite elements
4. Computational bases for serendipity-type spaces
5. Extension to generic quads and hexes
The ‘Periodic Table of the Finite Elements’


Classification of many common conforming finite element types.

<table>
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<tr>
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<tr>
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<td>$dP_2$</td>
<td>$dP_2$</td>
<td>$dQ_2$</td>
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$\rightarrow$ Domains in $\mathbb{R}^2$ (top half) and in $\mathbb{R}^3$ (bottom half)

<table>
<thead>
<tr>
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</tbody>
</table>

Order 1, 2, 3 of error decay (going down columns)

<table>
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<tr>
<th>$k$</th>
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<th>$k$</th>
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<tbody>
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<td>...</td>
<td>$n$</td>
<td>$n$</td>
</tr>
</tbody>
</table>

Conformity type $k = 0, \ldots, n$ (going across a row)

Geometry types: Simplices (left half) and cubes (right half).
Classification of conforming methods

Conforming finite element method types can be broadly classified by three integers:

- \( n \rightarrow \) the spatial dimension of the domain
- \( r \rightarrow \) the order of error decay
- \( k \rightarrow \) the differential form order of the solution space

**Ex:** \( Q^-_1 \Lambda^2(\square_3) \) is an element for

- \( n = 3 \rightarrow \) domains in \( \mathbb{R}^3 \)
- \( r = 1 \rightarrow \) linear order of error decay
- \( k = 2 \rightarrow \) conformity in \( \Lambda^2(\mathbb{R}^3) \hookrightarrow H(\text{div}) \)

\( Q^-_1 \Lambda^2(\square_3) \) is part of the \( Q^- \) ‘column’ of elements,

- is defined on geometry \( \square_3 \) (i.e. a cube),
- has a 6 dimensional space of test functions,
- and has an associated set of 6 degrees of freedom

that are unisolvent for the test function space.
An abbreviated reading list (50 years of theory!)


**Nédélec**, “Mixed finite elements in \( \mathbb{R}^3 \),” *Numerische Mathematik*, 1980


**Nédélec**, “A new family of mixed finite elements in \( \mathbb{R}^3 \),” *Numerische Mathematik*, 1986


**Arnold, Awano** “Finite element differential forms on cubical meshes”, *Math Comp.*, 2013

**Arnold, Boffi, Bonizzoni** “Finite element differential forms on curvilinear meshes and their approximation properties,” *Numerische Mathematik*, 2014
**$H(\text{div}) / L^2$ mixed form of Poisson problem**

Derivation of a mixed method for the **Poisson** problem on a domain $\Omega \subset \mathbb{R}^3$:

Given $f : \Omega \to \mathbb{R}$, find a function $p \in H^2(\Omega)$ such that

$$\Delta p + f = 0, \quad \text{in } \Omega, \text{ + B.C.'s}$$

Writing this as a first order system: find $u \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

\begin{align*}
\text{div } u + f &= 0, \quad \text{in } \Omega, \\
(u, v) + (p, \text{div } v) &= 0,
\end{align*}

\text{(\partial \Omega \text{ conditions})} = 0

A **weak form** of these equations: find $u \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

\begin{align*}
(\text{div } u, w) + (f, w) &= 0, \quad \forall \ w \in L^2 \quad = \quad \Lambda^3(\Omega) \\
(u, v) + (p, \text{div } v) &= 0, \quad \forall \ v \in H(\text{div}) \quad = \quad \Lambda^2(\Omega)
\end{align*}

i.e. $v$, div $v \in L^2(\Omega)$

differential form notation

A conforming mixed **finite element** method: find $u_h \in \Lambda_h^2$ and $p \in \Lambda_h^3$ such that

\begin{align*}
(\text{div } u_h, w_h) + (f, w_h) &= 0 \quad \forall \ w_h \in \Lambda_h^3 \quad \subset \quad L^2(\Omega) \\
(u_h, v_h) + (p_h, \text{div } v_h) &= [\partial \Omega \text{ terms}] \quad \forall \ v_h \in \Lambda_h^2 \quad \subset \quad H(\text{div}) \\
(\partial \Omega \text{ conditions}) &= 0
\end{align*}
A conforming mixed method for Darcy Flow

Movement of a fluid through porous media modeled via **Darcy flow**:

Given $f$ and $g$, find pressure $p$ and velocity $u$ such that:

$$
\begin{align*}
    u + K \nabla p &= 0 \quad \text{in } \Omega, \\
    \text{div } u - f &= 0 \quad \text{in } \Omega, \\
    p &= g \quad \text{on } \partial \Omega,
\end{align*}
$$

where $K$ is a symmetric, uniformly positive definite tensor for permeability.

A **weak form** of these equations: find $u \in H(\text{div})$ and $p \in L^2(\Omega)$ such that

$$
\begin{align*}
    (K^{-1}u, v) - (p, \text{div } v) &= [\partial \Omega \text{ terms}] \quad \forall \ v \in H(\text{div}) \\
    (\text{div } u, w) - (f, w) &= 0 \quad \forall \ w \in L^2(\Omega) \\
    (\partial \Omega \text{ conditions}) &= 0
\end{align*}
$$

A conforming mixed **finite element** method: find $u_h \in \Lambda^2_h$ and $p \in \Lambda^3_h$ such that

$$
\begin{align*}
    (K^{-1}u_h, v_h) - (p_h, \text{div } v_h) &= [\partial \Omega \text{ terms}] \quad \forall \ v_h \in \Lambda^2_h \subset H(\text{div}) \\
    (\text{div } u_h, w_h) - (f, w_h) &= 0 \quad \forall \ w_h \in \Lambda^3_h \subset L^2(\Omega) \\
    (\partial \Omega \text{ conditions}) &= 0
\end{align*}
$$

Stable pairs of finite element spaces

\[(u_h, v_h) + (p_h, \text{div} \, v_h) = [\partial \Omega \text{ terms}] \quad \forall \, v_h \in \Lambda^2_h \subset H(\text{div})\]

\[(\text{div} \, u_h, w_h) + (f, w_h) = 0 \quad \forall \, w_h \in \Lambda^3_h \subset L^2(\Omega)\]

Given a selection for the finite element spaces \((\Lambda^2_h, \Lambda^3_h)\),

the method is said to be \textbf{stable} if the error in the computed solution \((u_h, p_h)\) is within a constant multiple \(C\) of the minimal \textit{possible} error. That is:

\[
\|u - u_h\|_{H(\text{div})} + \|p - p_h\|_{L^2} \leq C \left( \inf_{w \in \Lambda^2_h} \|u - w\|_{H(\text{div})} + \inf_{q \in \Lambda^3_h} \|p - q\|_{L^2} \right) \quad (*)
\]

Brezzi’s theorem establishes the following sufficient criteria for \((*)\):

\[
(w, w) \geq c \|w\|^2_{H(\text{div})}, \quad \forall \, w \in Z_h := \left\{ w \in \Lambda^2_h : (\text{div} \, w, q) = 0, \quad \forall \, q \in \Lambda^3_h \right\},
\]

\[
\sup_{w \in \Lambda^2_h} \frac{(\text{div} \, w, q)}{\|w\|_{H(\text{div})}} \geq c \|q\|_{L^2}, \quad \forall \, q \in \Lambda^3_h.
\]

If the pair \((\Lambda^2_h, \Lambda^3_h)\) satisfies these two criteria it is called a \textbf{stable pair}.

The importance of method selection

Vector Poisson problem
- Solutions by the standard non-mixed method (left) and by a mixed method (right).
- Only the second choice shows the correct behavior near the reentrant corner.

Poisson problem
- Solutions by two different choices for the finite element solution spaces in a mixed method.
- Only the second choice looks like the true solution: $x(1 - x)y(1 - y)$.

Examples and images borrowed from:
Stable pairs of elements for mixed methods

Picking elements from the table for a mixed method for the Poisson problem:

\[ \mathbb{P}_1 \times \mathbb{P}_1^\perp(\Delta) \subset H^1 \times H^1 \subset L^2 \]

Unstable method

\[ \mathbb{dP}_0 \times \mathbb{P}_1^\perp(\Delta) \]

\[ \Rightarrow \]

Provably stable method converges to
\[ u = x(1 - x)y(1 - y) \]

\[ \subset H(\text{div}) \subset L^2 \]

Example and images on right from:

Method selection and cochain complexes

\[ \begin{align*} \mathbb{RT}^0_{1/3} & \hookrightarrow \mathbb{P}_1^3(\mathbb{D}_2) \\ \mathbb{dP}_0 & \hookrightarrow \mathbb{P}_1^3(\mathbb{D}_2) \end{align*} \]

\[ \Rightarrow \]

\[ \text{Provably stable method converges to } u = x(1 - x)y(1 - y) \]

Stable pairs of elements for mixed Hodge-Laplacian problems are found by choosing consecutive spaces in compatible discretizations of the \( L^2 \) deRham Diagram.

\[ \begin{align*} H^1 & \xrightarrow{\nabla} H(\text{curl}) & \xrightarrow{\nabla \times} H(\text{div}) & \xrightarrow{\nabla \cdot} L^2 \\ \text{vector Poisson} & \quad \sigma & \quad \mu \quad \text{Maxwell's eqn's} & \quad h \\ \text{Darcy} / \text{Poisson} & \quad u & \quad \rho \end{align*} \]

Stable pairs are found from consecutive entries in a cochain complex.
Exact cochain complexes found in the table

Two kinds of families of cochain complexes on an tetrahedron in $\mathbb{R}^3$:

\begin{align*}
\mathcal{P}_r^{-} \Lambda^0 & \to \mathcal{P}_r^{-} \Lambda^1 \to \mathcal{P}_r^{-} \Lambda^2 \to \mathcal{P}_r^{-} \Lambda^3 \quad \text{‘trimmed’ polynomials} \\
\mathcal{P}_r \Lambda^0 & \to \mathcal{P}_{r-1} \Lambda^1 \to \mathcal{P}_{r-2} \Lambda^2 \to \mathcal{P}_{r-3} \Lambda^3 \quad \text{polynomials}
\end{align*}
Exact cochain complexes found in the table

On an $n$-simplex in $\mathbb{R}^n$:

\[
P_r^{-} \Lambda^0 \rightarrow P_r^{-} \Lambda^1 \rightarrow \cdots \rightarrow P_r^{-} \Lambda^{n-1} \rightarrow P_r^{-} \Lambda^n \quad \text{‘trimmed’ polynomials}
\]

\[
P_r \Lambda^0 \rightarrow P_{r-1} \Lambda^1 \rightarrow \cdots \rightarrow P_{r-n+1} \Lambda^{n-1} \rightarrow P_{r-n} \Lambda^n \quad \text{polynomials}
\]

On an $n$-dimensional cube in $\mathbb{R}^n$:

\[
Q_r^{-} \Lambda^0 \rightarrow Q_r^{-} \Lambda^1 \rightarrow \cdots \rightarrow Q_r^{-} \Lambda^{n-1} \rightarrow Q_r^{-} \Lambda^n \quad \text{tensor product}
\]

\[
S_r \Lambda^0 \rightarrow S_{r-1} \Lambda^1 \rightarrow \cdots \rightarrow S_{r-n+1} \Lambda^{n-1} \rightarrow S_{r-n} \Lambda^n \quad \text{serendipity}
\]

The ‘minus’ spaces proceed across rows of the PToFE ($r$ is fixed) while the ‘regular’ spaces proceed along diagonals ($r$ decreases).

Mysteriously, the degree of freedom count for mixed methods from the $P_r^{-}$ spaces is smaller than those from the $P_r$ spaces, while the opposite is true for the $Q_r^{-}$ and $S_r$ spaces.
Outline

1. The “Periodic Table of the Finite Elements”
2. How to find new finite elements by counting
3. Trimmed serendipity finite elements
4. Computational bases for serendipity-type spaces
5. Extension to generic quads and hexes
### Counting boundary and interior DoFs of $\mathcal{P}_r^- \Lambda^k$

<table>
<thead>
<tr>
<th>faces, edges, and, vertices</th>
<th>$\mathcal{P}_1^- \Lambda^0(\Delta_3)$</th>
<th>$\mathcal{P}_1^- \Lambda^1(\Delta_3)$</th>
<th>$\mathcal{P}_1^- \Lambda^2(\Delta_3)$</th>
<th>$\mathcal{P}_1^- \Lambda^3(\Delta_3)$</th>
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<tbody>
<tr>
<td>interior</td>
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<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>total</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>1</td>
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<table>
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<tr>
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<tr>
<td>total</td>
<td>10</td>
<td>20</td>
<td>15</td>
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Identifying an alternating sum pattern

### Table 1

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<th>$P_1^{-} \Lambda^0(\Delta_3)$</th>
<th>$P_1^{-} \Lambda^1(\Delta_3)$</th>
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<tr>
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<tr>
<td>total</td>
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### Table 2

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### Counting DoFs of $Q_r^− \Lambda^k$

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<th>$Q_1^{-} \Lambda^2(\square_3)$</th>
<th>$Q_1^{-} \Lambda^3(\square_3)$</th>
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</thead>
<tbody>
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<td>12</td>
<td>6</td>
<td>0</td>
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</tr>
<tr>
<td>interior</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>total</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td>1</td>
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<th>$Q_2^{-} \Lambda^2(\square_3)$</th>
<th>$Q_2^{-} \Lambda^3(\square_3)$</th>
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<td>1</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>-1</td>
</tr>
<tr>
<td>total</td>
<td>27</td>
<td>54</td>
<td>36</td>
<td>8</td>
<td>1</td>
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</tbody>
</table>
Predicting DoFs of $S^r_\Lambda^k$

How big would a “minimal dimension” cochain complex on cubes be?

Expect to recover $Q_1^\Lambda^k$ in lowest order case:

<table>
<thead>
<tr>
<th></th>
<th>$S_1^\Lambda^0(\square_3)$</th>
<th>$S_1^\Lambda^1(\square_3)$</th>
<th>$S_1^\Lambda^2(\square_3)$</th>
<th>$S_1^\Lambda^3(\square_3)$</th>
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<td>12</td>
<td>6</td>
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<tr>
<td>interior</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>total</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

For $r > 1$, we must have a constant multiple of DoFs per edge or face, and we have expected dimensions (by other reasoning) for $S_2^\Lambda^0$ and $S_2^\Lambda^3$:

<table>
<thead>
<tr>
<th></th>
<th>$S_2^\Lambda^0(\square_3)$</th>
<th>$S_2^\Lambda^1(\square_3)$</th>
<th>$S_2^\Lambda^2(\square_3)$</th>
<th>$S_2^\Lambda^3(\square_3)$</th>
<th>$\pm$ sum</th>
</tr>
</thead>
<tbody>
<tr>
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<td>$12e_1 + 6f_1$</td>
<td>$6f_2$</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>interior</td>
<td>0</td>
<td>$i_1$</td>
<td>$i_2$</td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>total</td>
<td>20</td>
<td>$12e_1 + 6f_1 + i_1$</td>
<td>$6f_2 + i_2$</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Also expect $e_1 = 2$ since this would augment the DoFs per edge by 1 from $r = 1$ case.
### Actual DoFs of $S_r^{-k}(r = 1, 2)$

<table>
<thead>
<tr>
<th></th>
<th>$S_1^{-0}(\Box_3)$</th>
<th>$S_1^{-1}(\Box_3)$</th>
<th>$S_1^{-2}(\Box_3)$</th>
<th>$S_1^{-3}(\Box_3)$</th>
<th>± sum</th>
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</thead>
<tbody>
<tr>
<td>boundary</td>
<td>8</td>
<td>12</td>
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<td>2</td>
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<td>1</td>
<td>-1</td>
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<tr>
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<td>8</td>
<td>12</td>
<td>6</td>
<td>1</td>
<td>1</td>
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<th>$S_2^{-3}(\Box_3)$</th>
<th>± sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>boundary</td>
<td>20</td>
<td>36</td>
<td>18</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>interior</td>
<td>0</td>
<td>0</td>
<td>3</td>
<td>4</td>
<td>-1</td>
</tr>
<tr>
<td>total</td>
<td>20</td>
<td>36</td>
<td>21</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>
### Actual DoFs of $S_r^{-} \Lambda^k (r = 2, 3)$

#### $S_2^{-} \Lambda^0 (\Box_3)$
- Boundary: 20
- Interior: 0
- Total: 20

#### $S_2^{-} \Lambda^1 (\Box_3)$
- Boundary: 36
- Interior: 0
- Total: 36

#### $S_2^{-} \Lambda^2 (\Box_3)$
- Boundary: 18
- Interior: 3
- Total: 21

#### $S_2^{-} \Lambda^3 (\Box_3)$
- Boundary: 0
- Interior: 4
- Total: 4

#### $\pm$ sum
- Total: 2

---

#### $S_3^{-} \Lambda^0 (\Box_3)$
- Boundary: 32
- Interior: 0
- Total: 32

#### $S_3^{-} \Lambda^1 (\Box_3)$
- Boundary: 66
- Interior: 0
- Total: 66

#### $S_3^{-} \Lambda^2 (\Box_3)$
- Boundary: 36
- Interior: 9
- Total: 45

#### $S_3^{-} \Lambda^3 (\Box_3)$
- Boundary: 0
- Interior: 10
- Total: 10

#### $\pm$ sum
- Total: 2
1. The “Periodic Table of the Finite Elements”

2. How to find new finite elements by counting

3. Trimmed serendipity finite elements

4. Computational bases for serendipity-type spaces

5. Extension to generic quads and hexes
A new column for the PToFE: the **trimmed serendipity** elements.

\[ S^r_{\Lambda^k(□_n)} \]

denotes approximation order \( r \), subset of \( k \)-form space \( \Lambda^k(\Omega) \), use on meshes of \( n \)-dim’l cubes.

Defined for any \( n \geq 1, 0 \leq k \leq n, r \geq 1 \)

Identical or analogous properties to all the other columns in the table.

The advantage of the \( S^r_{\Lambda^k} \) spaces is that they have fewer degrees of freedom for mixed methods than their tensor product and serendipity counterparts.
The polynomial space of $S_r^{-}\Lambda^k$

$S_r^{-}\Lambda^k(\square_n)$ is a space of differential $k$-forms whose coefficients are polynomials in $\mathbb{R}^n$.

$$S_r^{-}\Lambda^k = \mathcal{P}_r^{-}\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_r\Lambda^{k-1}$$

Polynomial coefficients in each summand:

- $\mathcal{P}_r^{-}\Lambda^k$ : anything up to degree $r - 1$ and some degree $r$
- $\mathcal{J}_r\Lambda^k$ : certain polynomials whose degree is between $r+1$ and $r+n-k-1$
- $d\mathcal{J}_r\Lambda^{k-1}$ : certain polynomials whose degree is between $r$ and $r+n-k-2$

The “regular” serendipity space has an analogous decomposition:

$$S_r\Lambda^k = \mathcal{P}_r\Lambda^k \oplus \mathcal{J}_r\Lambda^k \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}$$

This decomposition provides a direct sum into some precise but elaborate subspaces:

$$\mathcal{J}_r\Lambda^k(\mathbb{R}^n) := \sum_{l \geq 1} \kappa \mathcal{H}_{r+l-1,l}\Lambda^{k+1}(\mathbb{R}^n),$$

where

$$\mathcal{H}_{r,l}\Lambda^k(\mathbb{R}^n) := \{\omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \mid \text{ldeg } \omega \geq l\},$$

where $\text{ldeg}(x^{\alpha}dx_{\sigma}) := \#\{i \in \sigma^* : \alpha_i = 1\}$.
The degrees of freedom of $S_r^{-} \Lambda^k$

The degrees of freedom associated to a $d$-dimensional sub-face $f$ of an $n$-dimensional cube $\square_n$ are (for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$):

$$u \mapsto \int_f (\text{tr}_f u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)-1} \Lambda^{d-k}(f) \oplus d\mathcal{H}_{r-2(d-k)+1} \Lambda^{d-k-1}(f),$$

These degrees of freedom are unisolvent for $S_r^{-} \Lambda^k(\square_n)$.

The direct sum decomposition of the indexing space gives one way to count the dimension precisely:

$$\begin{align*}
\underbrace{\mathcal{P}_{r-2(d-k)-1} \Lambda^{d-k}(f)}_{\text{indexing space for } S_{r-1} \Lambda^k(f)} & \oplus \underbrace{d\mathcal{H}_{r-2(d-k)+1} \Lambda^{d-k-1}(f)}_{\text{subspace of } \mathcal{H}_{r-2(d-k)} \Lambda^{d-k}(f)}
\end{align*}$$
Dimension count and comparison

Formula for counting degrees of freedom of $S_r^{-} \Lambda^k(\square_n)$:

$$\min\{n, \lfloor r/2 \rfloor + k\} \sum_{d=k}^{n-r-d} 2^{n-d} \binom{n}{d} \left( \binom{r-d+2k-1}{d-k} \binom{r-d+k-1}{d-k} + \binom{r-d+2k}{k} \binom{r-d+k-1}{d-k-1} \right)$$

<table>
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<th>r=1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>112</td>
<td>216</td>
<td>392</td>
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<td>5</td>
<td>15</td>
<td>35</td>
<td>70</td>
<td>126</td>
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</tbody>
</table>
Key properties of the trimmed serendipity spaces

\[ Q_r^{-} \Lambda^0 \to Q_r^{-} \Lambda^1 \to \cdots \to Q_r^{-} \Lambda^{n-1} \to Q_r^{-} \Lambda^n \]  \hspace{1cm} \text{tensor product}

\[ S_r \Lambda^0 \to S_{r-1} \Lambda^1 \to \cdots \to S_{r-n+1} \Lambda^{n-1} \to S_{r-n} \Lambda^n \]  \hspace{1cm} \text{serendipity}

\[ S_r^{-} \Lambda^0 \to S_r^{-} \Lambda^1 \to \cdots \to S_r^{-} \Lambda^{n-1} \to S_r^{-} \Lambda^n \]  \hspace{1cm} \text{trimmed serendipity}

Subcomplex: \[ dS_r^{-} \Lambda^k \subset S_r^{-} \Lambda^{k+1} \]

Exactness: The above sequence is exact.
\[ \text{i.e. the image of incoming map = kernel of outgoing map} \]

Inclusion: \[ S_r \Lambda^k \subset S_r^{-} \Lambda^k \subset S_{r+1} \Lambda^k \]

Trace: \[ \text{tr}_f S_r^{-} \Lambda^k (\mathbb{R}^n) \subset S_r^{-} \Lambda^k (f), \quad \text{for any} \ (n-1)\text{-hyperplane} \ f \ \text{in} \ \mathbb{R}^n \]

Special cases:
\[ S_r^{-} \Lambda^0 = S_r \Lambda^0 \]
\[ S_r^{-} \Lambda^n = S_{r-1} \Lambda^n \]
\[ S_r^{-} \Lambda^k + dS_{r+1} \Lambda^{k-1} = S_r \Lambda^k. \]

Replace ‘S’ by ‘\( \mathcal{P} \)’ \( \sim \rightarrow \) key properties about the first two columns for \( \mathcal{P}_r^{-} \Lambda^k \) and \( \mathcal{P}_r \Lambda^k \)!
Mixed Method dimension comparison 1

Mixed method for Darcy problem: \[ \mathbf{u} + K \nabla p = 0 \]
\[ \text{div} \, \mathbf{u} - f = 0 \]

We compare degree of freedom counts among the three families for use on meshes of affinely-mapped squares or cubes, when a conforming method with (at least) order \( r \) decay in the approximation of \( p, \mathbf{u}, \) and \( \text{div} \, \mathbf{u} \) is desired.

### Total # of degrees of freedom on a square \((n = 2)\):

| \( r \) | \( |Q_r^- \Lambda^1| + |Q_r^- \Lambda^2| \) | \( |S_r \Lambda^1| + |S_{r-1} \Lambda^2| \) | \( |S_r^- \Lambda^1| + |S_r^- \Lambda^2| \) |
|---|---|---|---|
| 1 | 4+1 = 5 | 8+1 = 9 | 4+1 = 5 |
| 2 | 12+4 = 16 | 14+3 = 17 | 10+3 = 13 |
| 3 | 24+9 = 33 | 22+6 = 28 | 17+6 = 23 |

### Total # of degrees of freedom on a cube \((n = 3)\):

| \( r \) | \( |Q_r^- \Lambda^2| + |Q_r^- \Lambda^3| \) | \( |S_r \Lambda^2| + |S_{r-1} \Lambda^3| \) | \( |S_r^- \Lambda^2| + |S_r^- \Lambda^3| \) |
|---|---|---|---|
| 1 | 6+1 = 7 | 18+1 = 19 | 6+1 = 7 |
| 2 | 36+8 = 44 | 39+4 = 43 | 21+4 = 25 |
| 3 | 108+27 = 135 | 72+10 = 82 | 45+10 = 55 |
Mixed Method dimension comparison 2

Mixed method for Darcy problem: \[ \mathbf{u} + K \nabla p = 0 \]
\[ \text{div} \mathbf{u} - f = 0 \]

The number of interior degrees of freedom is reduced from tensor product, to serendipity, to (trimmed) serendipity:

**# of interior degrees of freedom on a square (n = 2):**

| r | \( |Q_r^{-} \Lambda_0^1| + |Q_r^{-} \Lambda_0^2| \) | \( |S_r \Lambda_0^1| + |S_{r-1} \Lambda_0^2| \) | \( |S_r^{-} \Lambda_0^1| + |S_r^{-} \Lambda_0^2| \) |
|---|-----------------|-----------------|-----------------|
| 1 | 0+1 = 1         | 0+1 = 1         | 0+1 = 1         |
| 2 | 4+4 = 8         | 2+3 = 5         | 2+3 = 5         |
| 3 | 12+9 = 21       | 6+6 = 12        | 5+6 = 11        |

**# of interior degrees of freedom on a cube (n = 3):**

| r | \( |Q_r^{-} \Lambda_0^2| + |Q_r^{-} \Lambda_0^3| \) | \( |S_r \Lambda_0^2| + |S_{r-1} \Lambda_0^3| \) | \( |S_r^{-} \Lambda_0^2| + |S_r^{-} \Lambda_0^3| \) |
|---|-----------------|-----------------|-----------------|
| 1 | 0+1 = 1         | 0+1 = 1         | 0+1 = 1         |
| 2 | 12+8 = 20       | 3+4 = 7         | 3+4 = 7         |
| 3 | 54+27 = 81      | 12+10 = 22      | 9+10 = 19       |
Mixed Method dimension comparison 3

Mixed method for Darcy problem:
\[ u + K \nabla p = 0 \]
\[ \text{div} \ u - f = 0 \]

Assuming interior degrees of freedom could be dealt with efficiently (e.g. by static condensation), trimmed serendipity elements still have the fewest DoFs:

| # of interface (edge) degrees of freedom on a square \( n = 2 \): |
|---|---|---|
| \( r \) | \( |Q_r \Lambda^1(\partial\square_2)| \) | \( |S_r \Lambda^1(\partial\square_2)| \) | \( |S^-_r \Lambda^1(\partial\square_2)| \) |
| 1 | 4 | 8 | 4 |
| 2 | 8 | 12 | 8 |
| 3 | 12 | 16 | 12 |

| # of interface (edge+face) degrees of freedom on a cube \( n = 3 \): |
|---|---|---|
| \( r \) | \( |Q_r \Lambda^2(\partial\square_3)| \) | \( |S_r \Lambda^2(\partial\square_3)| \) | \( |S^-_r \Lambda^2(\partial\square_3)| \) |
| 1 | 6 | 18 | 6 |
| 2 | 24 | 36 | 18 |
| 3 | 54 | 60 | 36 |
Outline

1. The “Periodic Table of the Finite Elements”
2. How to find new finite elements by counting
3. Trimmed serendipity finite elements
4. Computational bases for serendipity-type spaces
5. Extension to generic quads and hexes
Building a computational basis

**Goal:** find a computational basis for $S_1 \Lambda^1(\square_3)$:

- Must be $H(\text{curl})$-conforming
- Must have 24 functions, 2 associated to each edge of cube
- Must recover constant and linear approx. on each edge
- The approximation space contains:
  1. Any polynomial coefficient of at most linear order:
     \[
     \{1, x, y, z\} \times \{dx, dy, dz\} \rightarrow 12 \text{ forms}
     \]
  2. Certain forms with quadratic or cubic order coefficients shown in table at left \(\rightarrow 12\text{ forms}\)

- For constants, use “obvious” functions:
  \[
  \{(y \pm 1)(z \pm 1)dx, (x \pm 1)(z \pm 1)dy, (x \pm 1)(y \pm 1)dz\}
  \]
  e.g. \((y + 1)(z + 1)dx\) evaluates to zero on every edge
  except \(\{y = 1, z = 1\}\) where it is \(\equiv 4 \rightarrow\) constant approx.

Also, \((y + 1)(z + 1)dx\) can be written as a linear combo, by using the first three forms at left to get the \(yz\ \text{dx}\) term
For constant approx on edges, we used:
\{(y ± 1)(z ± 1)dx, (x ± 1)(z ± 1)dy, (x ± 1)(y ± 1)dz\}

• Guess for linear approx on edges:
\{x(y ± 1)(z ± 1)dx, y(x ± 1)(z ± 1)dy, z(x ± 1)(y ± 1)dz\}
e.g. \(x(y + 1)(z + 1)dx\) evaluates to 4x on \(\{y = 1, z = 1\}\).

• Unfortunately: \(x(y + 1)(z + 1)dx \notin S_1 \Lambda(\square_3)\)!

Why? \(x(y + 1)(z + 1)dx = (xyz + xy + xz + x)dx\)

but \(xyz\) \(dx\) only appears with other cubic order coefficients!

• Remedy: add \(dy\) and \(dz\) terms that vanish on all edges.
Computational basis element associated to \( \{ y = 1, z = 1 \} \):

\[
2x(y + 1)(z + 1) \, dx + (z + 1)(x^2 - 1) \, dy + (y + 1)(x^2 - 1) \, dz
\]

✓ Evaluates to 4x on \( \{ y = 1, z = 1 \} \) (linear approx.)
✓ Evaluates to 0 on all other edges
✓ Belongs to the space \( S_1 \Lambda(\Box_3) \):

\[
\begin{align*}
2xyz \, dx &+ x^2z \, dy &+ x^2y \, dz \\
2xy \, dx &+ x^2 \, dy &+ 0 \, dz \\
2xz \, dx &+ 0 \, dy &+ x^2 \, dz \\
2x \, dx &+ (-z - 1) \, dy &+ (-y - 1) \, dz \\
\end{align*}
\]

\( \leftarrow \) linear order

\( \Rightarrow \) summation and factoring yields the desired form)

There are 11 other such functions, one per edge. We have:

\[
S_1 \Lambda(\Box_3) = \underbrace{E_0 \Lambda^1(\Box_3)}_{\text{"obvious" basis for constant approx}} \oplus \underbrace{\tilde{E}_1 \Lambda^1(\Box_3)}_{\text{modified basis for linear approx}}
\]

\[
\text{dim } 24 = 12 + 12
\]
A complete table of computational bases

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<th>$k = 2$</th>
<th>$k = 3$</th>
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<tbody>
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<td>$\forall \Lambda^0(\square_3)$</td>
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<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td>$\bigoplus_{i=0}^{r-2} E_i \Lambda^0(\square_3)$</td>
<td>$\bigoplus_{i=0}^{r-1} E_i \Lambda^1(\square_3) \oplus \tilde{E}_r \Lambda^1(\square_3)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td>$\bigoplus_{i=4}^{r} F_i \Lambda^0(\square_3)$</td>
<td>$\bigoplus_{i=2}^{r} F_i \Lambda^1(\square_3) \oplus \hat{F}_r \Lambda^1(\square_3)$</td>
<td>$\bigoplus_{i=0}^{r-1} F_i \Lambda^2(\square_3) \oplus \tilde{F}_r \Lambda^2(\square_3)$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
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<td>$\bigoplus_{i=4}^{r} I_i \Lambda^1(\square_3)$</td>
<td>$\bigoplus_{i=2}^{r} I_i \Lambda^2(\square_3)$</td>
<td>$\bigoplus_{i=2}^{r} I_i \Lambda^3(\square_3)$</td>
</tr>
<tr>
<td>$S_r^− \Lambda^k$</td>
<td>$\forall \Lambda^0(\square_3)$</td>
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<tr>
<td></td>
<td>$\bigoplus_{i=0}^{r-2} E_i \Lambda^0(\square_3)$</td>
<td>$\bigoplus_{i=0}^{r-1} E_i \Lambda^1(\square_3)$</td>
<td>$\emptyset$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td></td>
<td>$\bigoplus_{i=4}^{r} F_i \Lambda^0(\square_3)$</td>
<td>$\bigoplus_{i=2}^{r} F_i \Lambda^1(\square_3) \oplus \tilde{F}_r \Lambda^1(\square_3)$</td>
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</tr>
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<td></td>
<td>$\bigoplus_{i=6}^{r} I_i \Lambda^0(\square_3)$</td>
<td>$\bigoplus_{i=4}^{r} I_i \Lambda^1(\square_3) \oplus \tilde{I}_r \Lambda^1(\square_3)$</td>
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<td>$\bigoplus_{i=2}^{r} I_i \Lambda^3(\square_3)$</td>
</tr>
</tbody>
</table>
Outline

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Serendipity elements struggle with reference mapping

Quadratic serendipity elements, mapped non-affinely, are only expected to converge at the rate of linear elements.


\[ \| u - u_h \|_{L^2} \quad \| \nabla (u - u_h) \|_{L^2} \]

- **Linear**
  \[ O(h^2) \quad O(h) \]

- **Quadratic**
  \[ O(h^2) \quad O(h) \]

- **Serendipity**
  \[ O(h^2) \quad O(h) \]

- **Quadratic tensor prod.**
  \[ O(h^3) \quad O(h^2) \]

Extensions to vector-valued and higher dimensions:

The virtual element technique

→ Analogues of conforming finite element spaces on squares can be treated as virtual elements.
→ Explicit basis functions are not needed to implement the method.
→ Related polygonal element methods (HHO, HDG, WG...) may offer similar approaches.

Beirão da Veiga, Brezzi, Marini, Russo “Serendipity face and edge VEM spaces”  
A finite element space on a general quadrilateral is built in two parts:

- Apply Piola mapping to functions associated to boundary of reference element.
- Define functions on the physical element corresponding to interior degrees of freedom in a way that ensures relevant polynomial approximation properties.

Recent advances in hex-dominant meshing

- A hex-dominant mesh with $\approx 1.3$ million cells, including $\approx 1$ million hexahedra.
- Re-meshed from a mesh of $\approx 10$ million tetrahedra.

FEniCS primarily supports simplicial elements

deal.ii primarily supports quad/hex elements

ALNÆS ET AL. “The FEniCS Project Version 1.5” Archive of Numerical Software 2015

Neither package supports (trimmed) serendipity elements yet. . .
. . . but that is likely to change in the near future!
Acknowledgments

Thanks for the invitation to speak!

Related Publications


Research Funding

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Slides and Pre-prints

http://math.arizona.edu/~agillette/