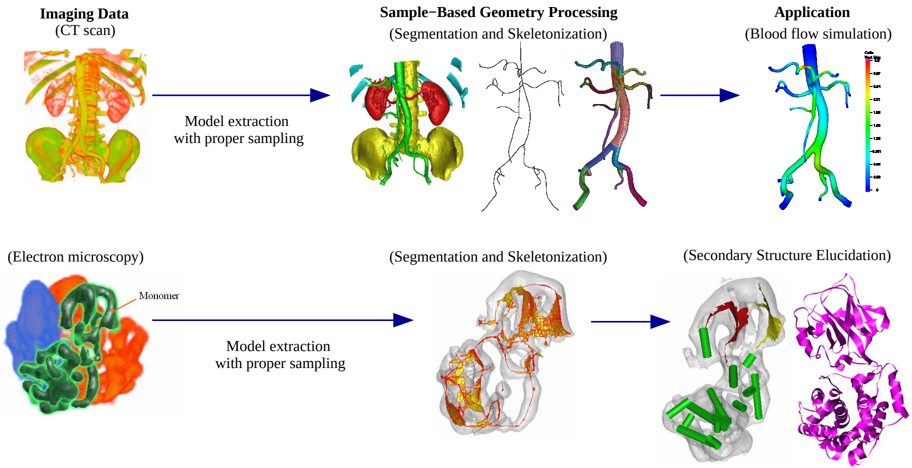
Critical Problems with Critical Points

Andrew Gillette

Department of Mathematics,
Institute of Computational Engineering and Sciences
University of Texas at Austin, Austin, Texas 78712, USA
http://www.math.utexas.edu/users/agillette

CVC Modeling Pipeline



Depending on the context, the presence of critical points can either aid in the algorithms of the pipeline or hinder them.

Surface Reconstruction

How critical are critical points?

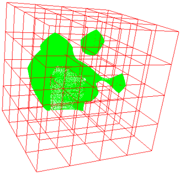
Surface Reconstruction

Input:

1. Rectilinear sampling of a bounded domain of an unknown scalar function $F : \mathbb{R}^3 \rightarrow \mathbb{R}$.
2. An isovalue $v \in \mathbb{R}$.
3. An interpolant for each sub-region.

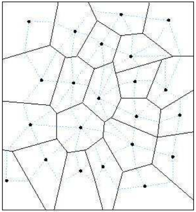
Output:

A mesh approximating the level set $F^{-1}(v)$.

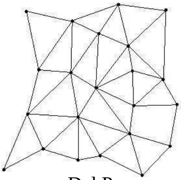


Can the critical point structure of the chosen interpolant aid in accurate surface reconstruction?

The Closed Ball Property



Vor P

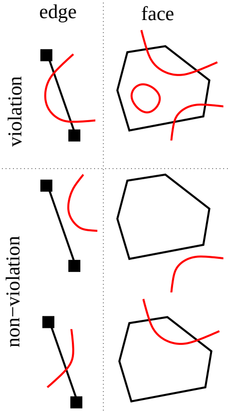


Del P

- Vor P = Voronoi diagram of pointset P.
- Del P = Delaunay diagram of pointset P.

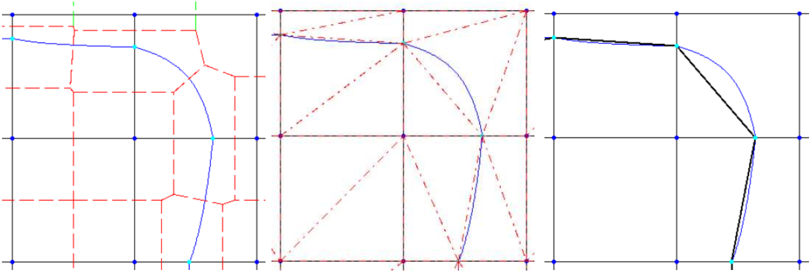
Closed ball property [ES,1997]

A Voronoi object V of dimension k satisfies the closed ball property iff $V \cap \Sigma = \emptyset$ or $V \cap \Sigma$ is homeomorphic to a closed ball of dimension $k - 1$.



Delaunay Conformity

$\mathrm{Del}\, P|_\Sigma$ = the set of Delaunay objects of $\mathrm{Del}\, P$ whose dual Voronoi objects have non-zero intersection with Σ.

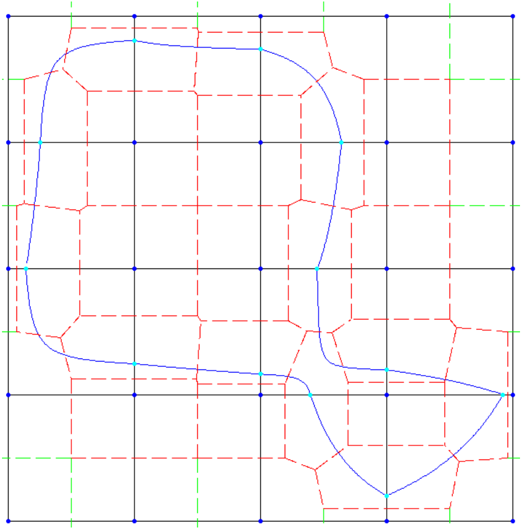


Theorem [ES,1997]

If Σ intersects each Voronoi object of $\mathrm{Vor}\, P$ transversally and $\mathrm{Vor}\, P$ satisfies the closed ball property, then $\mathrm{Del}\, P|_\Sigma$ is homeomorphic to Σ.

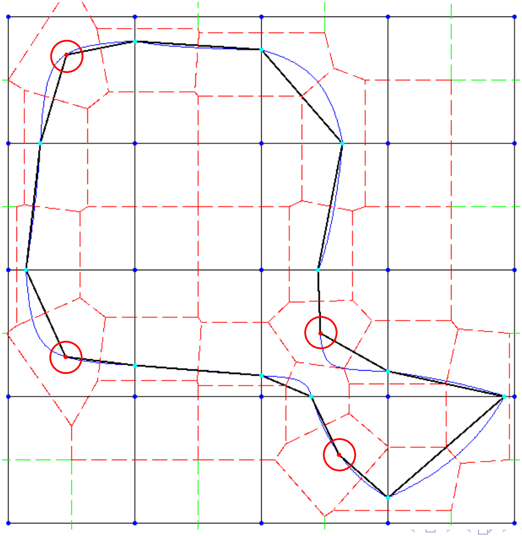
Reconstruction Algorithm [GGB,2007]

Compute the Voronoi diagram of Edge (E) and Grid (G) points.



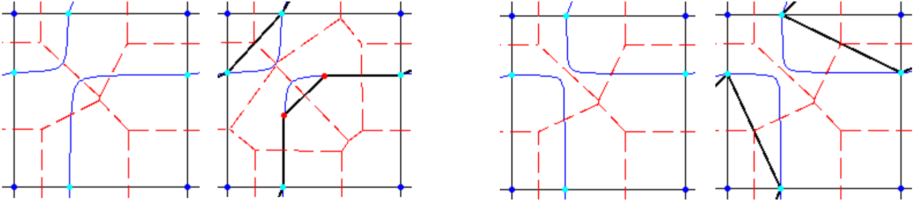
Reconstruction Algorithm [GGB,2007]

Insert new sample points (N) repeatedly until $\text{Vor}(E \cup G \cup N)$ satisfies the closed ball property.



Critical Points of the Interpolant

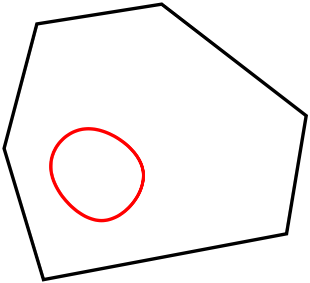
In the 2D bilinear case, if the closed ball property is violated, then there is a critical point in the pixel. The converse, however, is not true:

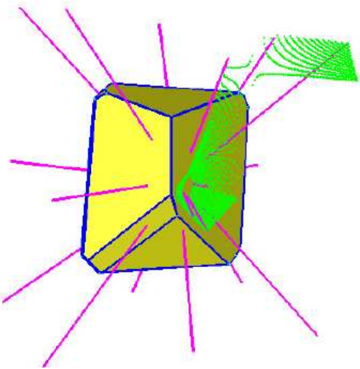


- Can we explain the sensitivity of the closed ball property to the presence of critical points?
- How does this analysis extend to higher order interpolants and to the 3D case?

Critical Points of the Surface

Since our surface is not C^1 continuous, the closed ball property can be violated in computationally difficult ways.





The problems become more severe for higher order interpolants, e.g. if the interpolant allows for an entire surface component within a single voxel.

(Un)Stable Manifold Computation

Critical points are our friends. . . if we can find them.

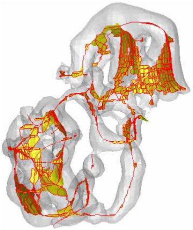
(Un)Stable Manifold Computation

Input:

1. A differentiable manifold M.
2. A smooth function $f : M \longrightarrow \mathbb{R}$.

Output:

The stable and unstable manifolds of f.



Questions:

- When can we solve this problem analytically?
- How accurate is our combinatorial approximation?

(Un)Stable Manifold Computation

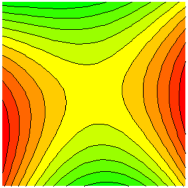
A point $m \in M$ is a **critical point** of f iff df_m is the zero map. It is **non-degenerate** iff the Hessian at m is non-singular.

Morse Lemma: f exhibits quadratic behavior in the neighborhood of a non-degenerate critical point m. If M is a compact 3-manifold, this implies for some neighborhood U of m:

$$f(x,y,z) = f(m) \pm x^2 \pm y^2 \pm z^2 \quad \forall (x,y,z) \in U$$

The **index** of a critical point is the number of minus signs in the above equation.

(Un)Stable Manifold Computation



An **integral path** of f is a maximal path $r : \mathbb{R} \longrightarrow M$ with

$$r'(t) = \nabla f|_{r(t)} \quad \forall t \in \mathbb{R}$$

Integral paths begin at a critical point of index k and end at a critical point of index $k+1$.

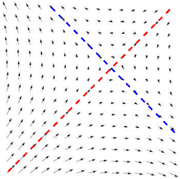
The **stable (unstable) manifold** of a non-degenerate critical point $m \in M$ is the union of m and the images of all integral paths on M terminating (beginning) at m.

The stable manifold of an index k critical point has boundary the stable manifold of an index $k-1$ critical point.

(Un)Stable Manifold Computation

$$f(x,y) := Axy + Bx + Cy + D$$

Solve the system of differential equations:



$$x'(t) = Ay(t) + B$$
$$y'(t) = Ax(t) + C$$

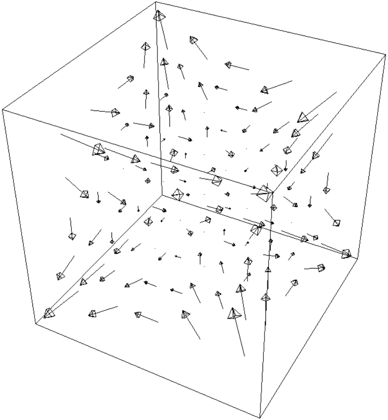
Parametric form of the integral curve through (x_0, y_0):

$$x(t) = -\frac{C}{A} + \left(\frac{x_0 + y_0}{2} + \frac{B+C}{2A}\right)e^{At} + \left(\frac{x_0 - y_0}{2} + \frac{C-B}{2A}\right)e^{-At}$$
$$y(t) = -\frac{B}{A} + \left(\frac{x_0 + y_0}{2} + \frac{B+C}{2A}\right)e^{At} + \left(\frac{y_0 - x_0}{2} + \frac{B-C}{2A}\right)e^{-At}$$

(Un)Stable Manifold Computation

$$f(x,y,z) = Axyz + Bx + Cy + Dz + E$$

Solve the (non-linear) system of differential equations:



$$\begin{aligned} x'(t) &= Ay(t)z(t) + B \\ y'(t) &= Az(t)x(t) + C \\ z'(t) &= Ax(t)y(t) + D \end{aligned}$$

$A = -1, B = C = D = 1$

(Un)Stable Manifold Computation

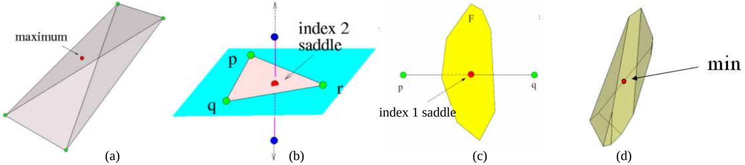
Given a compact surface Σ smoothly embedded in $\mathbb{R}^3$, the distance function h_Σ is a Morse function:

$$h_\Sigma : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad x \mapsto \inf_{p \in \Sigma} ||x - p||$$

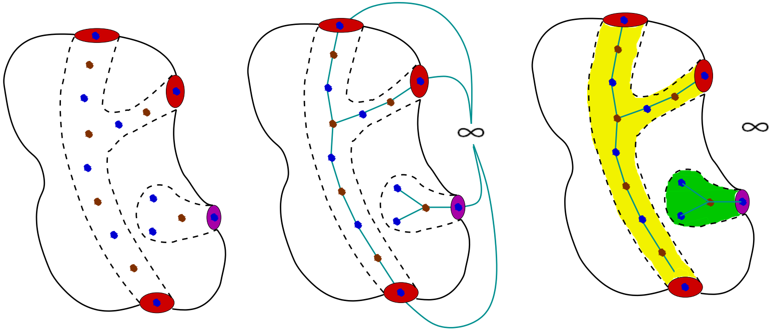
For a combinatorial reduction, we sample Σ by a finite point set P

$$h_P : \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad x \mapsto \min_{p \in P} ||x - p||$$

The critical points of h_P lie at the intersection of a Voronoi object and its dual Delaunay object.

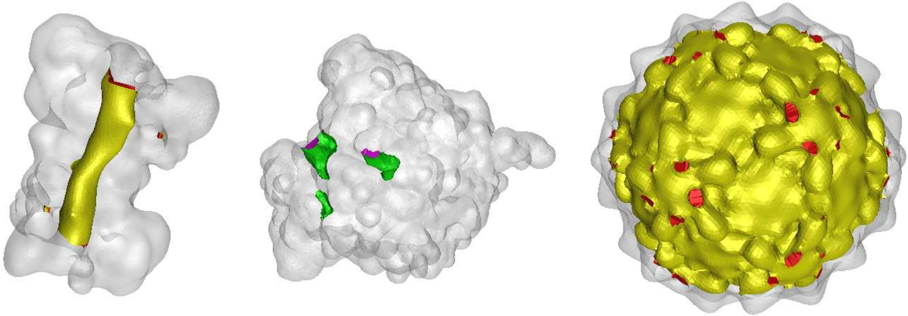


Using adjacencies provided by the Voronoi / Delaunay diagrams, we can detect tunnels, pockets, and voids of an embedded surface.



(a) (b) (c)

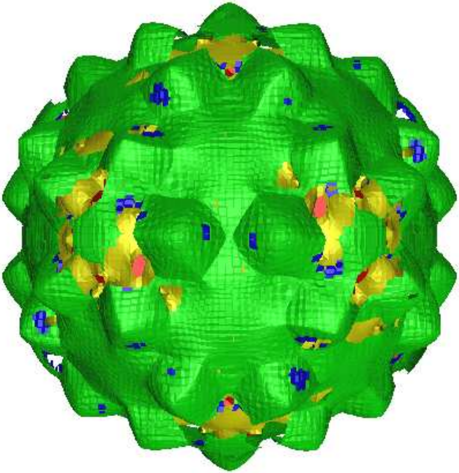
(Un)Stable Manifold Computation [BGG,2007]



1MAG; mAchE; nodavirus.

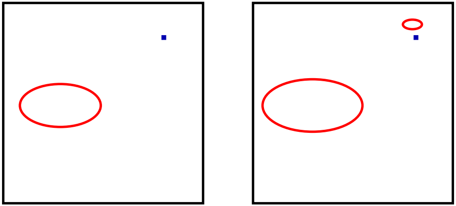
(Un)Stable Manifold Computation

Using a similar algorithm on primal space, we can also detect "thin" regions of a surface.



This method only works for a *fixed* isosurface. If we change the isosurface just slightly, the distance function may change dramatically.

Consider the value of the distance function at the blue point in each picture:



We need to find a way to update the distance function efficiently.

References

C. Bajaj, S. Goswami, and A. Gillette *Topology Based Selection and Curation of Level Sets* TopoInVis, 2007.

H. Edelsbrunner and N. Shah *Triangulating Topological Spaces* Intl. Journal of Comput. Geom. and Appl, 1997.

S. Goswami, A. Gillette and C. Bajaj *Efficient Delaunay Mesh Generation From Sampled Scalar Functions* IMR, 2007.

For a copy of these slides, please visit

http://www.math.utexas.edu/users/agillette/