Gradient bounds for Wachspress coordinates on polytopes

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joint work with M. Floater and N. Sukumar

Slides and more info at:

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Anatomy of a finite element error estimate

Goal: Approximate an unknown function $u : P \to \mathbb{R}$ where P is a polygon in $\mathbb{R}^2$.

Approach: Use barycentric coordinates $\{\phi_v\}$ associated to vertices $\{\mathbf{v}\}$ of P.

An error estimate can then be derived:

$$\underbrace{\left|\left| u - \sum_{\mathbf{v}} u(\mathbf{v})\phi_{\mathbf{v}} \right|\right|_{H^1(P)}}_{\substack{\text{approximation error} \\ \text{in value and derivative}}} \leq \underbrace{C_{\textit{proj-}\phi} \quad C_{\textit{proj-linear}} \quad \operatorname{diam}(P)}_{\text{constants}} \quad \underbrace{|u|_{H^2(P)}}_{\substack{\text{2nd order} \\ \text{oscillation} \\ \text{in } u}}$$

$C_{\textit{proj-}\phi} \approx$ operator norm of projection $u \longmapsto \sum_{\mathbf{v}} u(\mathbf{v})\phi_{\mathbf{v}}$

$C_{\textit{proj-linear}} \approx$ operator norm of projection $u \longmapsto$ linear polynomials on P

$\operatorname{diam}(P) =$ diameter of polygon P.

Key question for polygonal finite element methods

What geometrical properties of P can cause $C_{\textit{proj-}\phi}$ to be large?

Mathematical characterization

Problem statement

Given a simple convex d-dimensional polytope P, define

$$\Lambda := \sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla \phi_{\mathbf{v}}(\mathbf{x})|$$

where $\phi_{\mathbf{v}}$ are **generalized barycentric coordinates** on P.

Find upper and lower bounds on Λ in terms of geometrical properties of P.

Remark: It can be shown that

$$C_{\textit{proj-}\phi} = 1 + C_S(1 + \Lambda)$$

where C_S is the Sobolev embedding constant satisfying $||u||_{C^0(\overline{P})} \leq C_S\, ||u||_{H^k(P)}$ independent of $u \in H^k(P)$, provided that $k > d/2$.

Hence, bounds on Λ help us characterize when $C_{\textit{proj-}\phi}$ is large.

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1. Upper and lower bounds for simple convex polytopes
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The generalized barycentric coordinate approach

Let P be a convex polytope with vertex set V. We say that

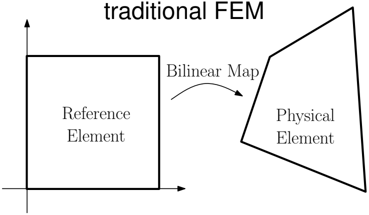
$$\phi_{\mathbf{v}} : P \to \mathbb{R} \quad \text{are} \quad \textbf{generalized barycentric coordinates (GBCs)} \text{ on } P$$

if they satisfy $\quad \phi_{\mathbf{v}} \geq 0$ on P and $L = \sum_{\mathbf{v}\in V} L(\mathbf{v_v})\phi_{\mathbf{v}}, \quad \forall\, L : P \to \mathbb{R}$ linear.

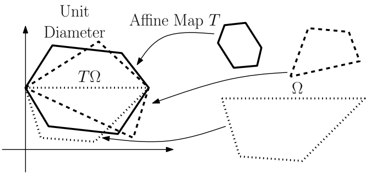
Familiar properties are implied by this definition:

$$\underbrace{\sum_{\mathbf{v}\in V} \phi_{\mathbf{v}} \equiv 1}_{\text{partition of unity}} \qquad \underbrace{\sum_{\mathbf{v}\in V} \mathbf{v}\phi_{\mathbf{v}}(\mathbf{x}) = \mathbf{x}}_{\text{linear precision}} \qquad \underbrace{\phi_{\mathbf{v}_i}(\mathbf{v}_j) = \delta_{ij}}_{\text{interpolation}}$$

traditional FEM



family of **GBC** reference elements



The triangular case

$$\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v} \in V} |\nabla \phi_{\mathbf{v}}(\mathbf{x})|$$

If P is a triangle, Λ can be large when P has a large interior angle.

→ This is often called the **maximum angle condition** for finite elements.

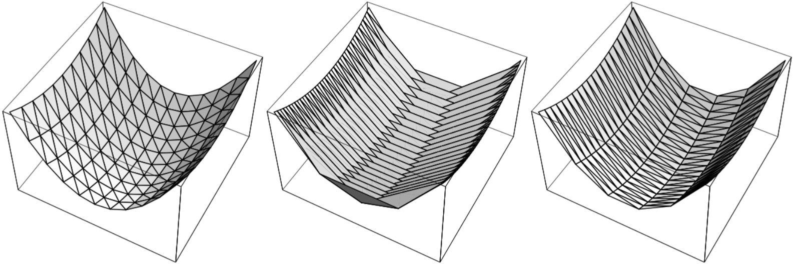
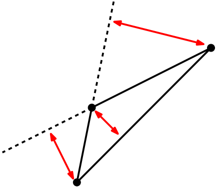


Figure from: SHEWCHUK *What is a good linear element?* Int'l Meshing Roundtable, 2002.

BABUŠKA, AZIZ *On the angle condition in the finite element method*, SIAM J. Num. An., 1976.

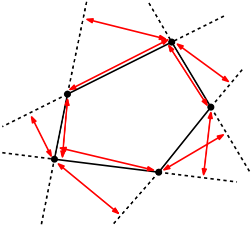
JAMET *Estimations d'erreur pour des éléments finis droits presque dégénérés*, ESAIM:M2AN, 1976.

Motivation



Observe that on triangles of fixed diameter:

$$|\nabla \phi_{\mathbf{v}}| \text{ large} \iff \text{interior angle at } \mathbf{v} \text{ is large}$$
$$\iff \text{the altitude "at } \mathbf{v}\text{" is small}$$



For Wachspress coordinates, we generalize to polygons:

$$|\nabla \phi_{\mathbf{v}}| \text{ large} \iff \text{the "altitude" at } \mathbf{v} \text{ is small}$$

and then to **simple** polytopes.

*A **simple** d-dimensional polytope has exactly d faces at each vertex.*

Given a simple convex d-dimensional polytope P, let

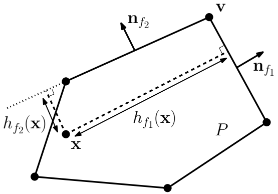
$$h_* := \text{minimum distance from a vertex to a hyper-plane of a non-incident face.}$$

Then $$\sup_{\mathbf{x}\in P} \sum_{\mathbf{v}\in V} |\nabla \phi_{\mathbf{v}}(\mathbf{x})| =: \Lambda \text{ is large} \iff h_* \text{ is small}$$

Upper bound for simple convex polytopes

Theorem [Floater, G., Sukumar, 2014]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\phi_{\mathbf{v}}$ be **generalized Wachspress coordinates**. Then $\Lambda \leq \dfrac{2d}{h_*}$ where $h_* = \min_f \min_{\mathbf{v} \notin f} \text{dist}(\mathbf{v}, f)$



$$\mathbf{p}_f(\mathbf{x}) := \frac{\mathbf{n}_f}{h_f(\mathbf{x})} = \text{normal to face } f\text{, scaled by the reciprocal of the distance from } \mathbf{x} \text{ to } f$$

$$w_{\mathbf{v}}(\mathbf{x}) := \det(\mathbf{p}_{f_1}(\mathbf{x}), \cdots, \mathbf{p}_{f_d}(\mathbf{x}))$$

$= $ volume formed by the d vectors $\{\mathbf{p}_{f_i}(\mathbf{x})\}$ for the d faces incident to $\mathbf{v}$

The **generalized Wachspress coordinates** are defined by $\phi_{\mathbf{v}}(\mathbf{x}) := \dfrac{w_{\mathbf{v}}(\mathbf{x})}{\sum_{\mathbf{u}} w_{\mathbf{u}}(\mathbf{x})}$

WACHSPRESS, *A Rational Finite Element Basis*, 1975.

JU, SCHAEFER, WARREN, DESBRUN, *A geometric construction of coordinates for convex polyhedra using polar duals* in *Geometry Processing 2005*, Eurographics, 2005.

Proof sketch for upper bound

To prove: $\sup_{\mathbf{x}} \sum_{\mathbf{v}} |\nabla \phi_{\mathbf{v}}(\mathbf{x})| =: \Lambda \leq \frac{2d}{h_*}$ where $h_* := \min_f \min_{\mathbf{v} \notin f} h_f(\mathbf{v})$.

1. Bound $|\nabla \phi_{\mathbf{v}}|$ by summations over faces incident and not incident to $\mathbf{v}$.

$$|\nabla \phi_{\mathbf{v}}| \leq \phi_{\mathbf{v}} \sum_{f \in F_{\mathbf{v}}} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \phi_{\mathbf{u}}\right) + \phi_{\mathbf{v}} \sum_{f \notin F_{\mathbf{v}}} \frac{1}{h_f} \left(\sum_{\mathbf{u} \in f} \phi_{\mathbf{u}}\right)$$

2. Summing over $\mathbf{v}$ gives a constant bound.

$$\sum_{\mathbf{v}} |\nabla \phi_{\mathbf{v}}| \leq 2 \sum_{f \in F} \frac{1}{h_f} \left(1 - \sum_{\mathbf{u} \in f} \phi_{\mathbf{u}}\right) \left(\sum_{\mathbf{u} \in f} \phi_{\mathbf{u}}\right)$$

3. Write $h_f(\mathbf{x})$ using $\phi_{\mathbf{v}}$ (possible since h_f is linear) and derive the bound.

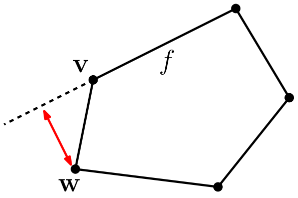
$$\Lambda \leq 2 \sum_{f \in F} \left(\sum_{\mathbf{u} \in f} \phi_{\mathbf{u}}\right) \frac{1}{h_*} = 2 \sum_{\mathbf{v} \in V} |\{f \ : \ f \ni \mathbf{v}\}| \phi_{\mathbf{v}} \frac{1}{h_*} = \frac{2d}{h_*}$$

Lower bound for polytopes

Theorem [Floater, G., Sukumar, 2014]

Let P be a simple convex polytope in $\mathbb{R}^d$ and let $\phi_{\mathbf{v}}$ be **any** generalized barycentric coordinates on P. Then

$$\frac{1}{h_*} \leq \Lambda$$



Proof sketch:

1. Show that $h_* = h_f(\mathbf{w})$, for some particular face f of P and vertex $\mathbf{w} \notin f$.
2. Let $\mathbf{v}$ be the vertex in f closest to $\mathbf{w}$. Show that
$$|\nabla \phi_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})}$$
3. Conclude the result, since
$$\Lambda \geq |\nabla \phi_{\mathbf{w}}(\mathbf{v})| = \frac{1}{h_f(\mathbf{w})} = \frac{1}{h_*}$$

Outline

Upper and lower bounds on polytopes

For a polytope $P \subset \mathbb{R}^d$, define $\Lambda := \sup_{\mathbf{x} \in P} \sum_{\mathbf{v}} |\nabla \phi_{\mathbf{v}}(\mathbf{x})|$.

simple convex polytope in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{2d}{h_*}$
d-simplex in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+1}{h_*}$
hyper-rectangle in $\mathbb{R}^d$	$\frac{1}{h_*}$	$\leq$	Λ	$\leq$	$\frac{d+\sqrt{d}}{h_*}$
regular k-gon in $\mathbb{R}^2$	$\frac{2(1+\cos(\pi/k))}{h_*}$	$\leq$	Λ	$\leq$	$\frac{4}{h_*}$

Note that $\lim_{k \to \infty} 2(1 + \cos(\pi/k)) = 4$, so the bound is **sharp** in $\mathbb{R}^2$.

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Relation to previous results

Prior work proved an upper bound on $|\nabla\phi_{\mathbf{v}}|$ for Wachspress coordinates on polygons:

$$|\nabla\phi_{\mathbf{v}}| \leq \frac{\pi^2}{2}\left(\frac{4}{(d_*)^4 \sin(\beta_*/2)\cos(\beta^*/2)\sin\beta^*}\right)^{2\beta^*/(\pi-\beta^*)}$$

where interior angles lie in $[\beta_*, \beta^*] \subset (0,\pi)$ and edges have length $\geq d_*$.

From this work, we have the simpler bound:

$$|\nabla\phi_{\mathbf{v}}| \leq \frac{4}{d_*(\sin\beta_*)(\sin\beta^*)}$$

A future project is to simplify existing bounds for the mean value coordinates.

G, Rand, Bajaj *Error Estimates for Generalized Barycentric Interpolation*
Advances in Computational Mathematics, 37:3, 417-439, 2012

Rand, G, Bajaj *Interpolation Error Estimates for Mean Value Coordinates*,
Advances in Computational Mathematics, 39:2, 327-347, 2013.

Outline

Matlab code for Wachspress coordinates on polygons

Input: The vertices $v_1, \ldots, v_n$ of a polygon and a point x

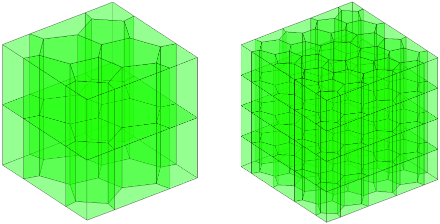
Output: Wachspress functions ϕ_i and their gradients $\nabla\phi_i$

```
function [phi dphi] = wachspress2d(v,x)
n = size(v,1);
w = zeros(n,1);
R = zeros(n,2);
phi = zeros(n,1);
dphi = zeros(n,2);

un = getNormals(v); % computes the outward unit normal to each edge

p = zeros(n,2);
for i = 1:n
    h = dot(v(i,:)  - x,un(i,:));
    p(i,:)  = un(i,:)  / h;
end

for i = 1:n
    im1 = mod(i-2,n) + 1;
    w(i) = det([p(im1,:);p(i,:)]);
    R(i,:)  = p(im1,:)  + p(i,:);
end

wsum = sum(w);
phi = w/wsum;

phiR = phi' * R;
for k = 1:2
    dphi(:,k) = phi .* (R(:,k) - phiR(:,k));
end
```

Matlab code for polygons *and polyhedra* (simple or non-simple) included in appendix of

FLOATER, G, SUKUMAR *Gradient bounds for Wachspress coordinates on polytopes*, SIAM J. Numerical Analysis, 2014.

Numerical results



h= 0.7071 h= 0.3955

→ We fix a sequence of polyhedral meshes where h denotes the maximum diameter of a mesh element.

→ $\exists\ \gamma > 0$ such that if any element from any mesh in the sequence is scaled to have diameter 1, the computed value of h_* will be $\geq \gamma$.

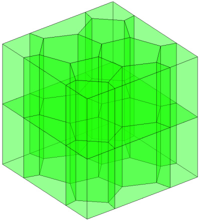
→ We solve the weak form of the Poisson problem:

$$\int_\Omega \nabla u \cdot \nabla w \, d\mathbf{x} = \int_\Omega f\ w \, d\mathbf{x}, \quad \forall w \in H_0^1(\Omega),$$

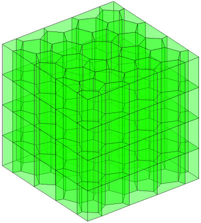
where $f(\mathbf{x})$ is defined so that the exact solution is $u(\mathbf{x}) = xyz(1-x)(1-y)(1-z)$.

→ Using Wachspress coordinates $\phi_\mathbf{v}$, the local stiffness matrix has entries of the form $\int_P \nabla \phi_\mathbf{v} \cdot \nabla \phi_\mathbf{w} \, d\mathbf{x}$, which we integrate by tetrahedralizing P and using a second-order accurate quadrature rule (4 points per tetrahedron).

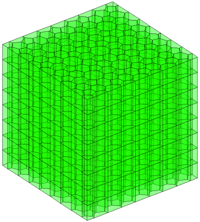
Numerical results



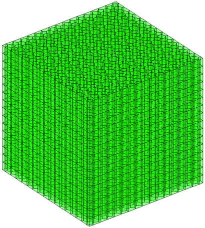
$h = 0.7071$



$h = 0.3955$



$h = 0.1977$



$h = 0.0989$

As expected, we observe optimal convergence convergence rates: quadratic in L^2 norm and linear in H^1 semi-norm.

Mesh	# of nodes	h	$\frac{\|\|u-u^h\|\|_{0,P}}{\|\|u\|\|_{0,P}}$	Rate	$\frac{\|u-u^h\|_{1,P}}{\|u\|_{1,P}}$	Rate
a	78	0.7071	2.0×10^{-1}	–	4.1×10^{-1}	–
b	380	0.3955	5.4×10^{-2}	2.28	2.1×10^{-1}	1.14
c	2340	0.1977	1.4×10^{-2}	1.96	1.1×10^{-1}	0.97
d	16388	0.0989	3.5×10^{-3}	1.99	5.4×10^{-2}	0.99
e	122628	0.0494	8.8×10^{-4}	**2.00**	2.7×10^{-2}	**0.99**

Acknowledgments

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Thanks for the invitation to speak!

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