Finite Element Clifford Algebra: A New Toolkit for Evolution Problems

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Motivation

Poisson's equation: Given f find $u(x)$ such that

$$\begin{cases} 0 = \Delta u + f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Heat equation: Given f and g, find $u(x,t)$ such that

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \subset \mathbb{R}^n, & \text{for } t > 0, \\ u = 0 & \text{on } \partial\Omega, & \text{for } t > 0, \\ u|_{t=0} = g & \text{in } \Omega \end{cases}$$

Finite element exterior calculus (FEEC) provides:

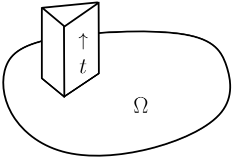
- abstract framework for analyzing numerical approximation of elliptic PDEs
- classification of stable finite element methods with optimal convergence rates

How can the FEEC framework be expanded to classify stable finite element methods for evolutionary PDEs?

Outline of approach

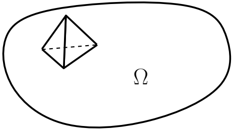
Two possible methods for extending Finite Element Exterior Calculus:

- **Semi-discrete:** Finite element method in space, ODE in time



domain:	$\Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$
solution basis:	$\phi_h\vert_{t=t_0} : \Omega \to \mathbb{R}$, for each $t_0 \in [0, T]$
error analysis:	FEEC + Bochner space theory

- **Fully discrete:** Finite element method in space ***and*** time



domain:	$\Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$
solution basis:	$\phi_h : \Omega \times [0, T] \to \mathbb{R}$
error analysis:	Finite Element Clifford Algebra

This talk: Initial results on semi-discrete approach + a preview of FECA

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Finite Element Exterior Calculus in context

Consider a mixed method for Poisson's problem on a domain $\Omega \subset \mathbb{R}^n$:

continuous $\quad \Delta u + f = 0, \quad u \in H^2$

mixed weak

$$
\begin{aligned}
(\text{div}\ \sigma, \phi) + (f, \phi) &= 0, \quad \forall \phi \in L^2 &= \Lambda^n \\
(\sigma, \omega) + (u, \text{div}\ \omega) &= 0, \quad \forall \omega \in H(\text{div}) &= \Lambda^{n-1}
\end{aligned}
$$

mixed FEM

$$
\begin{aligned}
(\text{div}\ \sigma_h, \phi_h) + (f, \phi_h) &= 0, \quad \forall \phi_h \in \Lambda_h^n &\subset L^2 \\
(\sigma_h, \omega_h) + (u_h, \text{div}\ \omega_h) &= 0, \quad \forall \omega_h \in \Lambda_h^{n-1} &\subset H(\text{div})
\end{aligned}
$$

Major Conclusions from FEEC

- The finite elements spaces Λ_h^{n-1} and Λ_h^{n-1} should be chosen from two classes of piecewise polynomial spaces, denoted $\mathcal{P}_r\Lambda_h^k$ and $\mathcal{P}_r^-\Lambda_h^k$
- If this choice is made in a compatible manner **implied by the exterior calculus structure**, then optimal *a priori* error estimates are guaranteed

Finite Element Exterior Calculus in context

Theorem [Arnold, Falk, Winther; Bulletin of AMS, 2010]

Assume the elliptic regularity estimate

$$||u||_{H^{s+2}} + ||\nabla u||_{H^{s+1}} + ||\sigma||_{H^{s+1}} + ||\text{div }\sigma||_{H^s} \leq c\,||f||_{H^s}$$

holds for $0 \leq s \leq s_{\max}$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^{n} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}^{-}\Lambda^{n}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}) \end{array} \right\}$$

Then for $0 \leq s \leq s_{\max}$, the following error estimates hold

$$||u - u_h||_{L^2} \leq \begin{cases} ch\,||f||_{L^2} & \text{if } \Lambda_h^n = \mathcal{P}_1^{-}\Lambda^n(\mathcal{T}), \\ ch^{2+s}\,||f||_{H^s} & \text{otherwise,} \qquad\qquad \text{if } s \leq r-1 \end{cases}$$

$$||\sigma_h - \sigma||_{L^2} \leq ch^{s+1}\,||f||_{H^s} \quad \text{if} \quad \begin{cases} \Lambda_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}, & s \leq r+1, \\ \Lambda_h^{n-1} = \mathcal{P}_{r+1}^{-}\Lambda^{n-1}, & s \leq r, \end{cases}$$

$$||\text{div }(\sigma_h - \sigma)||_{L^2} \leq ch^{s}\,||f||_{H^s}\,, \quad \text{if} \quad s \leq r+1.$$

Semi-discrete Mixed Formulation

Consider a mixed method for the heat equation on $\Omega \subset \mathbb{R}^n$ for $t \in I := [0, T]$.

continuous

$$\begin{aligned} u_t - \Delta u &= f, \\ u|_{t=0} &= g. \end{aligned}$$

mixed weak

$$\begin{aligned} (u_t, \phi) - (\text{div } \sigma, \phi) &= (f, \phi), & \forall \phi \in \Lambda^n, && t \in I, \\ (\sigma, \omega) + (u, \text{div } \omega) &= 0, & \forall \omega \in \Lambda^{n-1}, && t \in I, \\ u|_{t=0} &= g. \end{aligned}$$

mixed FEM

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div } \sigma_h, \phi_h) &= (f, \phi_h), & \forall \phi_h \in \Lambda_h^n, && t \in I, \\ (\sigma_h, \omega_h) + (u_h, \text{div } \omega_h) &= 0, & \forall \omega_h \in \Lambda_h^{n-1}, && t \in I, \\ u_h|_{t=0} &= g_h. \end{aligned}$$

linear system

$$\begin{aligned} AU_t - B\Sigma &= F \\ B^T U + D\Sigma &= 0 \end{aligned} \qquad \Rightarrow \; AU_t + BD^{-1}B^T U = F$$

Semi-discrete Error Bounds

Theorem [Thomée; Galerkin FEM for Parabolic Problems, 1997]

Fix $n = 2$ and set $\Lambda_h^2 :=$ discontinuous linear, $\Lambda_h^1 :=$ Raviart-Thomas elements.

- Let g_h be the solution to the elliptic problem with $f = -\Delta g$. Then for $t \geq 0$:

$$\begin{aligned} ||u_h(t) - u(t)||_{L^2} &\leq ch^2 \left(||u(t)||_{H^2} + \int_0^t ||u_t||_{H^2}\, ds \right), \\ ||\sigma_h(t) - \sigma(t)||_{L^2} &\leq ch^2 \left(||u(t)||_{H^3} + \left(\int_0^t ||u_t||_{H^2}^2\, ds \right)^{1/2} \right). \end{aligned}$$

- Homogeneous case ($f = 0$), g_h as above, $t \geq 0$:

$$||u_h(t) - u(t)||_{L^2} \leq ch^2 |g|_{H^2}, \quad \text{if } g \in \dot{H}^2,$$

$$||\sigma_h(t) - \sigma(t)||_{L^2} \leq ch^3 |g|_{H^3}, \quad \text{if } g \in \dot{H}^3.$$

- Homogeneous case ($f = 0$), $g_h :=$ orthogonal projection of g on to Λ_h^2, $t > 0$:

$$||u_h(t) - u(t)||_{L^2} \leq ch^2 t^{-1} ||g||_{L^2}$$

$$||\sigma_h(t) - \sigma(t)||_{L^2} \leq ch^2 t^{-3/2} ||g||_{L^2}$$

Note: These bounds are 'space-only' and restricted to the case $n = 2$.

Bochner spaces and norms

Our new error bounds will employ the theory of **Bochner spaces**

Definition

Let X be a Banach space and $I = (0, T)$. Define

$$C(I, X) := \{u : I \to X \mid u \text{ bounded and continuous}\}$$

Equip this space with the norm

$$||u||_{C(I,X)} := \sup_{t \in I} ||u(t)||_X .$$

The **Bochner space** $L^P(I, X)$ is defined to be the completion of $C(I, X)$ with respect to the norm:

$$||u||_{L^p(I,X)} := \left(\int_I ||u(t)||_X^p \, dt \right)^{1/p} .$$

We combine notations to get Bochner differential form spaces:

$$L^2\mathfrak{X}^k := L^2(I, L^2\Lambda^k(\Omega))$$

These are parametrized differential form spaces.

Outline

Bochner-FEEC Parabolic Error Estimates

We combine the FEEC and parabolic error estimates to derive the following.

Theorem [G, Holst, 2011]

Let $n \geq 2$ and fix $I := [0, T]$. Suppose regularity estimate

$$||u(t)||_{H^{s+2}} + ||\nabla u(t)||_{H^{s+1}} + ||\sigma(t)||_{H^{s+1}} + ||\text{div}\,\sigma(t)||_{H^s} \leq c\,||f(t)||_{H^s}$$

holds for $0 \leq s \leq s_{\max}$ and $t \in I$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \left\{ \begin{array}{c} \mathcal{P}_{r+1}^-\Lambda^n(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_r\Lambda^n(\mathcal{T}) \end{array} \right\}$$

Then for $0 \leq s \leq s_{\max}$ and g_h the solution to the elliptic problem we have

$$||u_h - u||_{L^2\mathfrak{X}^n} \leq \begin{cases} ch\left(||f||_{L^2(I,L^2)} + \sqrt{T}\,||f_t||_{L^1(I,L^2)}\right) & \text{if } \Lambda_h^n = \mathcal{P}_1^-\Lambda^n(\mathcal{T}) \\ ch^{2+s}\left(||f||_{L^2(I,H^s)} + \sqrt{T}\,||f_t||_{L^1(I,H^s)}\right) & \text{otherwise, if } s \leq r-1 \end{cases}$$

and. . .

Bochner-FEEC Paraoblic Error Estimates

Theorem [G, Holst, 2011]

Let $n \geq 2$ and fix $I := [0, T]$. Suppose regularity estimate

$$||u(t)||_{H^{s+2}} + ||\nabla u(t)||_{H^{s+1}} + ||\sigma(t)||_{H^{s+1}} + ||\text{div}\,\sigma(t)||_{H^s} \leq c\,||f(t)||_{H^s}$$

holds for $0 \leq s \leq s_{\max}$ and $t \in I$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \left\{ \begin{array}{c} \mathcal{P}_{r+1}^{-}\Lambda^{n}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r}\Lambda^{n}(\mathcal{T}) \end{array} \right\}$$

Then for $0 \leq s \leq s_{\max}$ and g_h the solution to the elliptic problem we have:

- If $\left\{ \begin{array}{c} \Lambda_h^{n-1} = \mathcal{P}_1\Lambda^{n-1}(\mathcal{T}),\ s \leq 1 \\ \text{or} \\ \Lambda_h^{n-1} = \mathcal{P}_1^{-}\Lambda^{n-1}(\mathcal{T}),\ s = 0 \end{array} \right\}$ and $\Lambda_h^n = \mathcal{P}_1^{-}\Lambda^n(\mathcal{T})$, then

$$||\sigma_h - \sigma||_{L^2\mathfrak{X}^{n-1}} \leq c\left(h^{1+s}\,||f||_{L^2(I,H^s)} + h\sqrt{T}\,||f_t||_{L^2(I,L^2)} \right)$$

- For any other choice of spaces, if $s \leq r - 1$,

$$||\sigma_h - \sigma||_{L^2\mathfrak{X}^{n-1}} \leq c\left(h^{1+s}\,||f||_{L^2(I,H^s)} + h^{2+s}\sqrt{T}\,||f_t||_{L^2(I,L^2)} \right)$$

Proof and Significance

Key idea of the proof:

$$\underbrace{||u(t) - u_h(t)||_{L^2}}_{\text{error between weak and semi-discrete}} \leq \underbrace{||u(t) - \tilde{u}_h(t)||_{L^2}}_{\text{error between weak and time-ignorant elliptic}} + \underbrace{||\tilde{u}_h(t) - u_h(t)||_{L^2}}_{\text{error between time-ignorant elliptic and semi-discrete}}$$

Significance of the error estimates

- These results give *a priori* estimates of convergence rates for the semi-discrete Galerkin FEM for the heat equation.
- By using the FEEC framework, we have classified choices of semi-discrete finite element spaces that guarantee optimal convergence rates.
- The results hold for arbitrary spatial dimension n, not just $n = 2$.
- For the homogeneous case ($f = 0$) with sufficiently regular g, we expect to find stronger error estimates akin to Thomée's.

G, HOLST, *Finite Element Exterior Calculus for Evolution Problems*, in preparation.

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The Bochner Complex

- FEEC theory studies discretizations of the L^2 **deRham complex**:

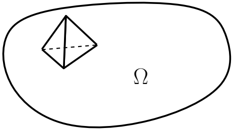
$$0 \longrightarrow H\Lambda^0 \xrightarrow[\text{(grad)}]{d_\Omega} H\Lambda^1 \xrightarrow{d_\Omega} \cdots \xrightarrow[\text{(div)}]{d_\Omega} H\Lambda^n \xrightarrow{d_\Omega} 0$$

- We can define a parametrized exterior derivative operator on Bochner spaces:

$$d : H\mathfrak{X}^k \to H\mathfrak{X}^{k+1} \quad \text{where} \quad (d\mu)(t) := d_\Omega(\mu(t)).$$

- This gives rise to a **Bochner domain complex**:

$$0 \longrightarrow H\mathfrak{X}^0 \xrightarrow{d} H\mathfrak{X}^1 \xrightarrow{d} \cdots \xrightarrow{d} H\mathfrak{X}^n \xrightarrow{d} 0$$



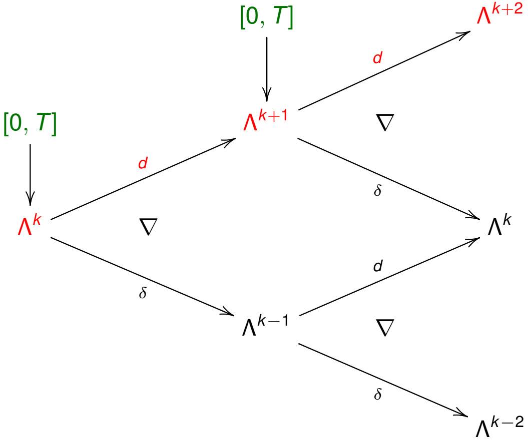
- For a 'fully discrete' method, we need an exterior derivative operator on **spacetime elements** which can distinguish spacelike and timelike dimensions.
- Such an operator needs the Lorentzian signature of basis elements - a tool available in Clifford Algebra (or Geometric Calculus) but not exterior calculus.

Beyond the deRham Complex. . .

- The ‘derivative’ operator ∇ in Clifford algebra is a formal sum of d and its adjoint:

$$\nabla := d + \delta$$

- The **deRham complex** appears as diagonals in a full ‘**Clifford complex**’
- The **Bochner complex** appears as parametrizations of these diagonals



Finite Element Clifford Algebra will study discretizations of this larger complex.

Questions?

- Slides and pre-prints available at http://ccom.ucsd.edu/∼agillette