Serendipity and Tensor Product Pyramid Finite Elements

THE UNIVERSITY OF ARIZONA

Andrew Gillette

Department of Mathematics, University of Arizona

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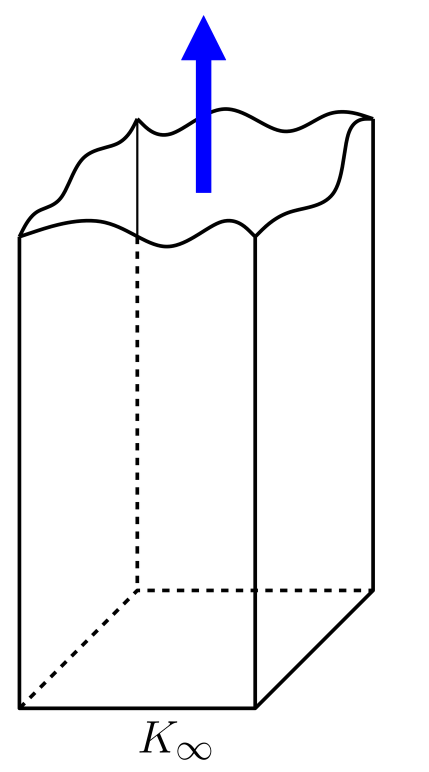
Pyramid Geometry



The **infinite pyramid** geometry is

$$K_\infty := \{\, (x,y,z)\in\mathbb{R}^3\cup\infty \ :\ 0\le x,y\le 1,\ 0\le z\le\infty \,\}.$$

The **reference pyramid** geometry is

$$\hat{K} := \{\, (\hat{x},\hat{y},\hat{z})\in\mathbb{R}^3 \ :\ 0\le \hat{x},\hat{y},\hat{z},\ \hat{x}\le 1-\hat{z},\ \hat{y}\le 1-\hat{z} \,\}.$$

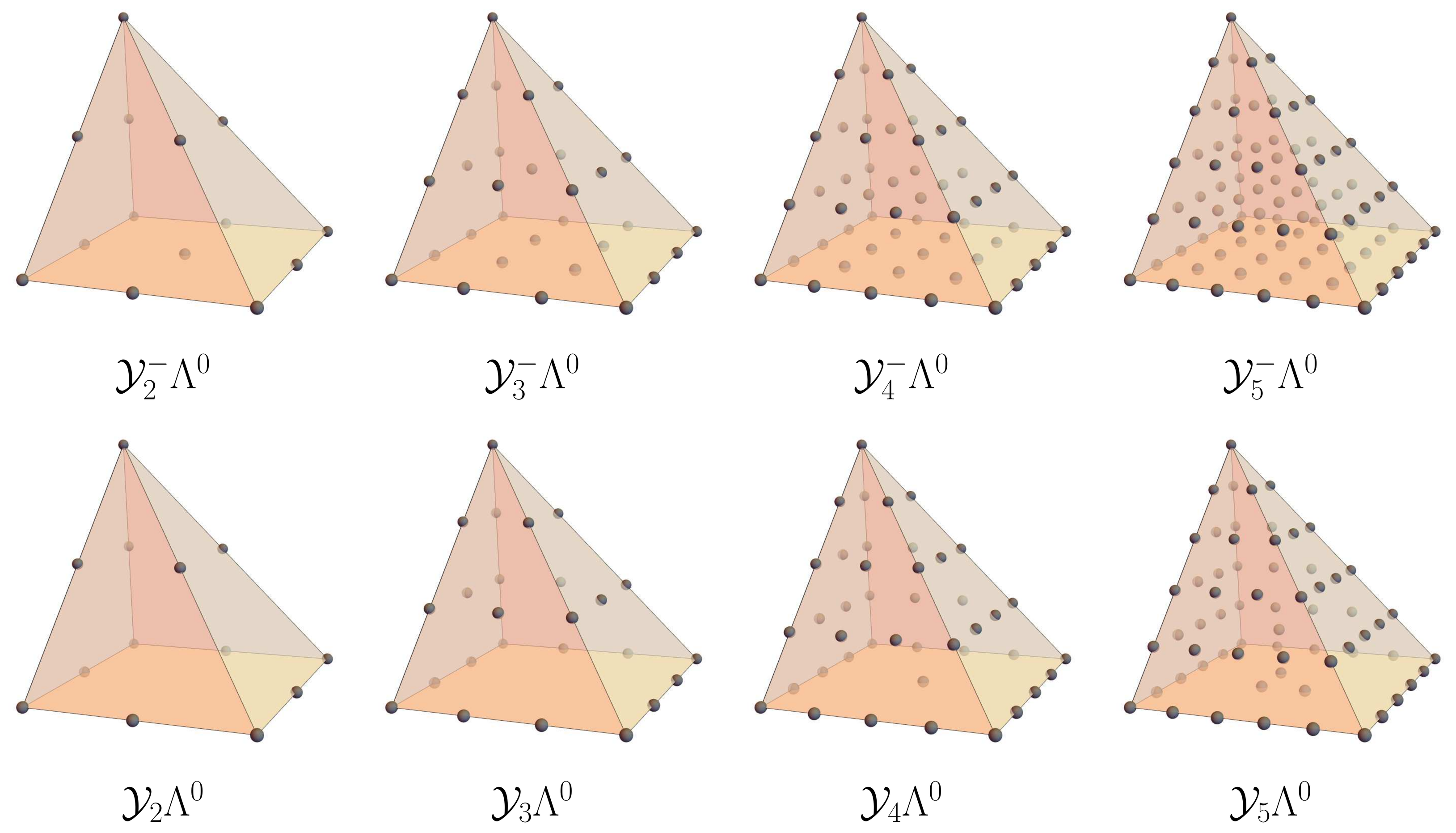
Define a bijective change of coordinates $\phi : K_\infty \to \hat{K}$ by

$$\phi(x,y,z) = \begin{cases} \left(\dfrac{\hat{x}}{1+\hat{z}}, \dfrac{\hat{y}}{1+\hat{z}}, \dfrac{\hat{z}}{1+\hat{z}}\right), & 0\le z<\infty \\ (0,0,1), & z=\infty \end{cases}$$

Given $u : \hat{K}\to\mathbb{R}$, the **pullback** of u to K^∞ by ϕ is

$$\phi^* u : K_\infty \to \mathbb{R} \quad \text{where} \quad (\phi^*u)(x,y,z) := u(\phi(x,y,z))$$

Degrees of freedom are associated to $\hat{K}$ as shown:



On the base of $\hat{K}$, $\mathcal{Y}_r^-\Lambda^0$ is identical to $\mathcal{Q}_r^-\Lambda^0(\square_2)$ (tensor product on a square), while $\mathcal{Y}_r\Lambda^0$ is identical $\mathcal{S}_r\Lambda^0(\square_2)$ (serendipity on a square).

Shape Functions

The **superlinear degree** of a monomial is defined by: $\quad \text{sldeg}\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right) := \sum_{\alpha_i\ne 1} \alpha_i.$

Define **shape functions on K_∞** as follows:

$$\mathcal{Q}_r^{[r,r]} := \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ :\ 0\le a,b\le j \right\}$$

$$\mathcal{S}_r^{[r,r]} := \bigoplus_{j=0}^{r} \left\{ \frac{x^a y^b}{(1+z)^j} \ :\ 0\le a,b\le j,\ \text{sldeg}(x^a y^b)\le j \right\}$$

Define the **lowest order bubble function** $\quad \mathtt{b} : K_\infty\to\mathbb{R} \quad$ by

$$\mathtt{b}(x,y,z) := \frac{x(1-x)y(1-y)z}{(1+z)^3} = \frac{x(1-x)y(1-y)}{(1+z)^2} - \frac{x(1-x)y(1-y)}{(1+z)^3}$$

Observe that $\mathtt{b}$ vanishes on ∂K_∞. Note that $\mathtt{b}\in\mathcal{Q}_r^{[r,r]} \iff r\ge 3$ and $\mathtt{b}\in\mathcal{S}_r^{[r,r]} \iff r\ge 5$.

Define **shape functions on $\hat{K}$** as those whose pullback by ϕ is a shape function on K_∞.

$$\phi\left(\mathcal{Q}_r^{[r,r]}\right) := \left\{ u:\hat{K}\to\mathbb{R} \ :\ \phi^*u\in\mathcal{Q}_r^{[r,r]} \right\}.$$

$$\phi\left(\mathcal{S}_r^{[r,r]}\right) := \left\{ u:\hat{K}\to\mathbb{R} \ :\ \phi^*u\in\mathcal{S}_r^{[r,r]} \right\}.$$

Dimension count

Comparison of dimension counts for various pyramid elements in the literature.

$r\to$	1	2	3	4	5	6	7	formula	key reference
$\dim\mathcal{Y}_r\Lambda^0$	5	13	25	42	65	95	133	$\dfrac{r^3+6r^2+23r}{6}$	[3]
$\dim\mathcal{Y}_r^-\Lambda^0$ $\dim\hat{P}_r$ $\dim\mathcal{R}_r^{(0)}$	5	14	30	55	91	140	204	$\dfrac{(r+1)(r+2)(2r+3)}{6}$	[3] [1] [5]
$\dim\mathcal{U}^{(0),r}$	5	15	37	77	141	235	365	r^3+3r+1	[2], [4]

Degrees of Freedom

Associate degrees of freedom to portions of the pyramid geometry as follows:

To each vertex $\mathbf{v}$,	$u\longmapsto u(\mathbf{v})$	
To each edge $\mathbf{e}$,	$u\longmapsto \int_{\mathbf{e}} (\text{tr}_{\mathbf{e}}\, u)\wedge q,$	$q\in P_{\mathbf{e}}$
To each triangular face $\triangle$,	$u\longmapsto \int_{\triangle} (\text{tr}_{\triangle}\, u)\wedge q,$	$q\in P_{\triangle}$
To the parallelogram face $\square$,	$u\longmapsto \int_{\square} (\text{tr}_{\square}\, u)\wedge q,$	$q\in P_{\square}$
To the three-dimensional interior $\mathtt{int}$,	$u\longmapsto \int_{\mathtt{int}} (\text{tr}_{\mathtt{int}}\, u)\wedge q,$	$q\in R_{\mathtt{int}}$

	$P_{\mathbf{v}}$	$P_{\mathbf{e}}$	$P_{\triangle}$	$P_{\square}$	$R_{\mathtt{int}}$
$\mathcal{Y}_r^-\Lambda^0$	$\mathbb{R}$	$\mathcal{P}_{r-2}\Lambda^1(\mathbf{e})$	$\mathcal{P}_{r-3}\Lambda^2(\triangle)$	$\mathcal{Q}_{r-1}^-\Lambda^2(\square)$	$\phi\left(\mathtt{b}\cdot\mathcal{Q}_{r-3}^{[r-3,r-3]}\right)\Lambda^3(\mathtt{int})$
$\mathcal{Y}_r\Lambda^0$	$\mathbb{R}$	$\mathcal{P}_{r-1}^-\Lambda^1(\mathbf{e})$	$\mathcal{P}_{r-2}^-\Lambda^2(\triangle)$	$\mathcal{P}_{r-4}\Lambda^2(\square)$	$\phi\left(\mathtt{b}\cdot\mathcal{S}_{r-5}^{[r-5,r-5]}\right)\Lambda^3(\mathtt{int})$

The notation for the $\mathtt{int}$ cases means

$$\phi\left(\mathtt{b}\cdot\mathcal{Q}_{r-3}^{[r-3,r-3]}\right)\Lambda^3(\mathtt{int}) := \text{span}\left\{ u\,dV \ :\ \phi^*u = \mathtt{b}q \text{ with } q\in\mathcal{Q}_{r-3}^{[r-3,r-3]} \right\},$$

where dV is the volume 3-form on $\hat{K}$. We count the degrees of freedom and find that

$$\dim\mathcal{Y}_r^-\Lambda^0 = \frac{1}{6}(2r^3+9r^2+13r+6) = \dim\mathcal{Q}_r^{[r,r]}$$

$$\dim\mathcal{Y}_r\Lambda^0 = \frac{1}{6}(r^3+6r^2+23r) = \dim\mathcal{S}_r^{[r,r]}$$

Theorem 1: The degrees of freedom for $\mathcal{Y}_r^-\Lambda^0$ are **unisolvent** for $\phi\left(\mathcal{Q}_r^{[r,r]}\right)$.

The degrees of freedom for $\mathcal{Y}_r\Lambda^0$ are **unisolvent** for $\phi\left(\mathcal{S}_r^{[r,r]}\right)$.

Theorem 2: The shape functions of order r on $\hat{K}$ **reproduce polynomials**, i.e.

$$\mathcal{P}_r(\mathbb{R}^3)\subset\phi\left(\mathcal{Q}_r^{[r,r]}\right) \quad\text{and}\quad \mathcal{P}_r(\mathbb{R}^3)\subset\phi\left(\mathcal{S}_r^{[r,r]}\right)$$

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