Trimmed Serendipity Finite Elements

Andrew Gillette and Tyler Kloefkorn

Department of Mathematics, University of Arizona

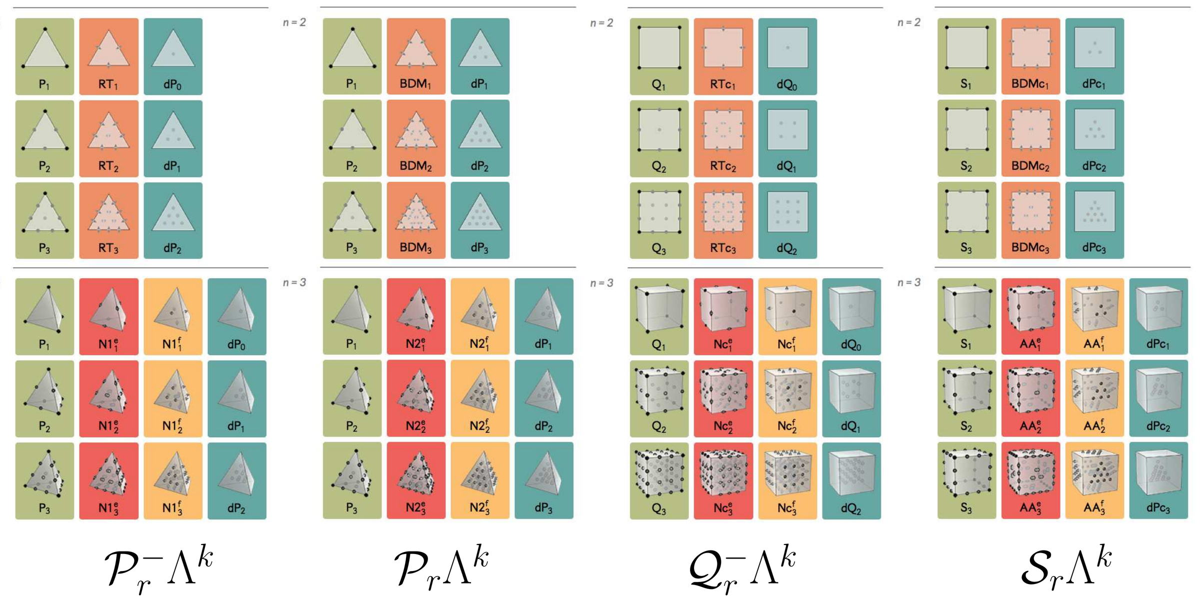
THE UNIVERSITY OF ARIZONA

supported in part by NSF Award 1522289

The Periodic Table of Finite Elements

→ The Periodic Table of Finite Elements identifies four families of conforming finite elements on simplices and cubes.

→ We describe a new family on cubes, called *Trimmed Serendipity Finite Elements*.



Definitions and Notation

The space of differential k-forms with polynomial coefficients of homogeneous degree r is denoted $\mathcal{H}_r\Lambda^k(\mathbb{R}^n)$. The exterior derivative d and Koszul operator κ are maps

$$d : \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \to \mathcal{H}_{r-1}\Lambda^{k+1}(\mathbb{R}^n) \qquad \kappa : \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \to \mathcal{H}_{r+1}\Lambda^{k-1}(\mathbb{R}^n).$$

The space of polynomial differential k-forms of degree at most r is

$$\mathcal{P}_r\Lambda^k(\mathbb{R}^n) := \bigoplus_{j=0}^{r} \mathcal{H}_j\Lambda^k(\mathbb{R}^n).$$

The space of trimmed polynomial differential k-forms of degree at most r is

$$\mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) := \mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n) + \kappa\mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n).$$

The linear degree of $x^\alpha dx_\sigma$ is defined to be ldeg$(x^\alpha dx_\sigma) := \#\{i \in \sigma^* \ : \ \alpha_i = 1\}$.

Example: ldeg$(xyz^2\, dx) = 1$, b/c y is the only variable with exponent 1 not appearing in dx.

Two key building blocks for both the serendipity and trimmed serendipity spaces are

$$\mathcal{H}_{r,l}\Lambda^k(\mathbb{R}^n) := \left\{\omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \ : \ \text{ldeg}\ \omega \geq l\right\}, \text{ and}$$

$$\mathcal{J}_r\Lambda^k(\mathbb{R}^n) := \sum_{l\geq 1} \kappa\mathcal{H}_{r+l-1,l}\Lambda^{k+1}(\mathbb{R}^n).$$

The serendipity differential k-forms of order r on an n-cube $\square_n$ are given by

$$\mathcal{S}_r\Lambda^k(\square_n) := \mathcal{P}_r\Lambda^k(\square_n) \oplus \mathcal{J}_r\Lambda^k(\square_n) \oplus d\mathcal{J}_{r+1}\Lambda^{k-1}(\square_n).$$

Trimmed Serendipity Finite Elements

We define trimmed serendipity differential k-forms of order r on an n-cube $\square_n$ by

$$\mathcal{S}_r^-\Lambda^k(\square_n) := \mathcal{S}_{r-1}\Lambda^k(\square_n) + \kappa\mathcal{S}_{r-1}\Lambda^{k+1}(\square_n).$$

Theorem: $\mathcal{S}_r^-\Lambda^k(\square_n) = \mathcal{P}_r^-\Lambda^k(\square_n) + \mathcal{J}_r\Lambda^k(\square_n) + d\mathcal{J}_r\Lambda^{k-1}(\square_n)$

The degrees of freedom for $\mathcal{S}_r^-\Lambda^k(\square_n)$ associated to a d-dimensional sub-face f of $\square_n$ are

$$u \longmapsto \int_f (\text{tr}_f\, u) \wedge q, \quad q \in \mathcal{P}_{r-2(d-k)-1}\Lambda^{d-k}(f) \ \oplus \ d\mathcal{H}_{r-2(d-k)+1}\Lambda^{d-k-1}(f),$$

for any $k \leq d \leq \min\{n, \lfloor r/2 \rfloor + k\}$.

Theorem: (Unisolvence) If $u \in \mathcal{S}_r^-\Lambda^k(\square_n)$ and all the degrees of freedom vanish, then $u \equiv 0$.

Dimension Count

Theorem: Fix $n, r \geq 1$ and $0 \leq k \leq n$. Then

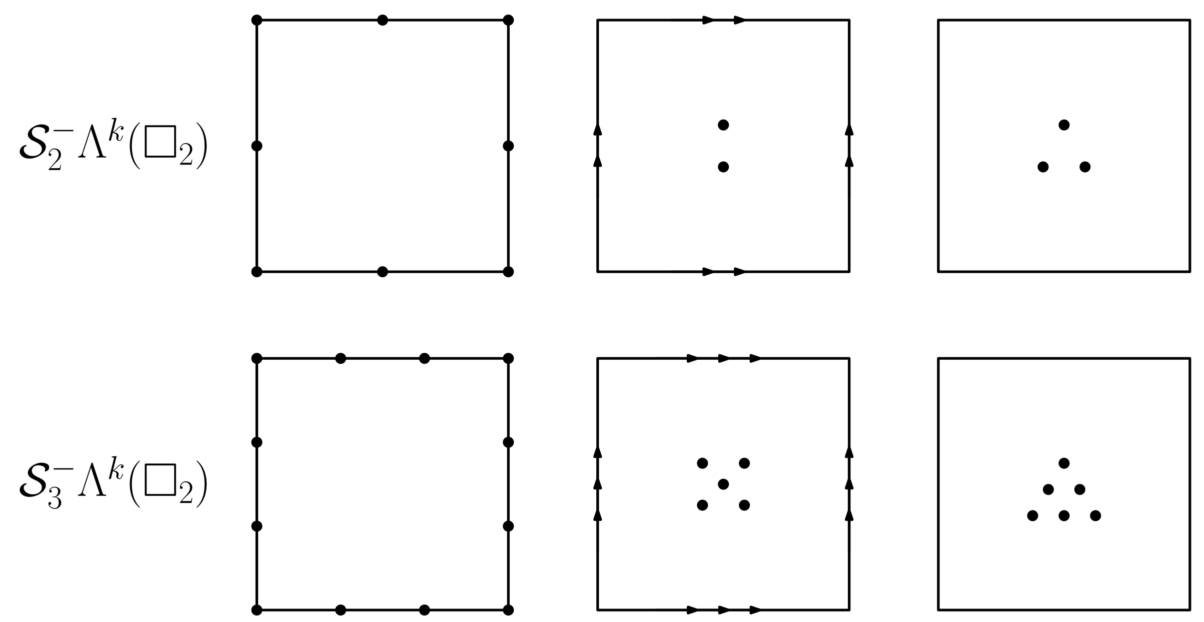
$$\dim \mathcal{S}_r^-\Lambda^k(\square_n) = \dim \mathcal{P}_r^-\Lambda^k(\square_n) + \dim \mathcal{J}_r\Lambda^k(\square_n) + \dim \ \mathcal{J}_r\Lambda^{k-1}(\square_n)$$

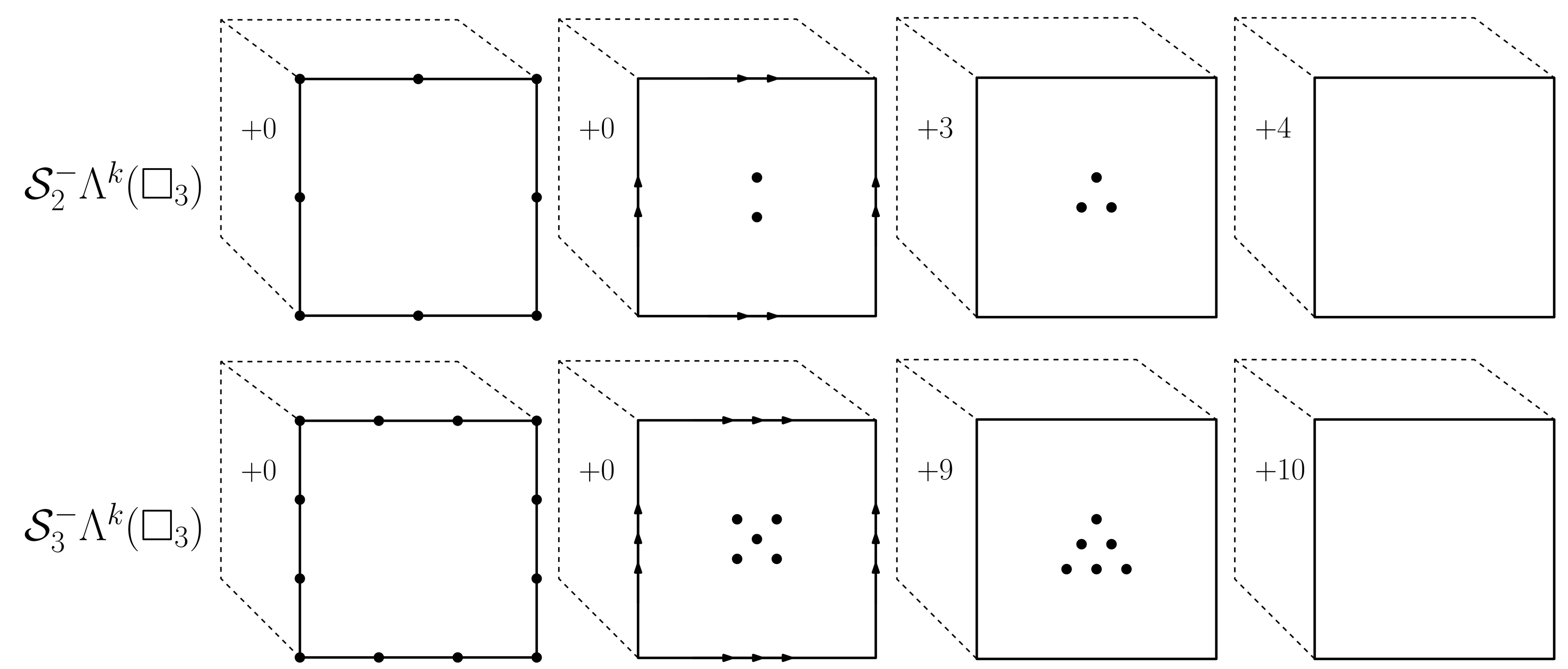
Further, each summand has a closed-form expression in terms of binomial coefficients depending only on n, k, and r.

		$\dim \mathcal{S}_r^-\Lambda^k(\square_n)$								$\dim \mathcal{S}_r\Lambda^k(\square_n)$						
	k	$r=1$	2	3	4	5	6	7	k	$r=1$	2	3	4	5	6	7
$n=1$	0	2	3	4	5	6	7	8	0	2	3	4	5	6	7	8
	1	1	2	3	4	5	6	7	1	2	3	4	5	6	7	8
$n=2$	0	4	8	12	17	23	30	38	0	4	8	12	17	23	30	38
	1	4	10	17	26	37	50	65	1	8	14	22	32	44	58	74
	2	1	3	6	10	15	21	28	2	3	6	10	15	21	28	36
$n=3$	0	8	20	32	50	74	105	144	0	8	20	32	50	74	105	144
	1	12	36	66	111	173	255	360	1	24	48	84	135	204	294	408
	2	6	21	45	82	135	207	301	2	18	39	72	120	186	273	384
	3	1	4	10	20	35	56	84	3	4	10	20	35	56	84	120

We find that $\dim \mathcal{S}_r^-\Lambda^k(\square_n) \leq \dim \mathcal{S}_r\Lambda^k(\square_n)$ with equality **only** when $k = 0$.

Element Diagrams and Related Work





→ Element diagrams indicate association of degrees of freedom to parts of the geometry, i.e. vertex, edge, face, or interior ($+X$).

→ The trace of $\mathcal{S}_r^-\Lambda^k(\square_n)$ on a face f is $\mathcal{S}_r^-\Lambda^k(f)$.

Vector element analogues

→ Arbogast and Correa (2015) define spaces $(\mathbf{V}^r_{AC}, W^r_{AC}) \subset H(\text{div}) \times L^2$. Interpreting these spaces on a reference square as differential forms via the flat operator, we find that $(\text{rot}\mathbf{V}^r_{AC}, W^r_{AC})$ is identical to $\left(\mathcal{S}_{r+1}^-\Lambda^1(\square_2), \mathcal{S}_{r+1}^-\Lambda^2(\square_2)\right)$ where rot means rotation by $\pi/2$.

→ Cockburn and Fu (2016) define sequences of spaces $S_{2,r}^{\square_2}$ on a reference square and $S_{2,r}^{\square_3}$ on a reference cube. Interpreting these sequences by the flat operator we recover the trimmed serendipity sequences:

$$\mathcal{S}_{r+1}^-\Lambda^0(\square_2) \to \mathcal{S}_{r+1}^-\Lambda^1(\square_2) \to \mathcal{S}_{r+1}^-\Lambda^2(\square_2)$$

$$\mathcal{S}_{r+1}^-\Lambda^0(\square_3) \to \mathcal{S}_{r+1}^-\Lambda^1(\square_3) \to \mathcal{S}_{r+1}^-\Lambda^2(\square_3) \to \mathcal{S}_{r+1}^-\Lambda^3(\square_3)$$

References

T. Arbogast, M. R. Correa. Two families of $H(div)$ mixed finite elements on quadrilaterals of minimal dimension. *SIAM J. Numerical Analysis*, 2016.

D. Arnold, G. Awanou. The serendipity family of finite elements. *Foundations of Computational Mathematics*, 2011.

D. Arnold, D. Boffi, F. Bonizzoni. Finite element differential forms on curvilinear cubic meshes. *Numerische Mathematik*, 2014.

D. Arnold, R. Falk, R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numerica*, 2006.

D. Arnold, R. Falk, R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bulletin of the AMS*, 2010.

D. Arnold, A. Logg. Periodic table of the finite elements. *SIAM News*, 2014. `femtable.org`

L. Beirão da Veiga, F. Brezzi, L. D. Marini, A. Russo. Serendipity face and edge VEM spaces. arXiv:1606.01048, 2016.

F. Brezzi, J. Douglas Jr, and L. D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numerische Mathematik*, 1985.

S. H. Christiansen. Foundations of finite element methods for wave equations of Maxwell type. *Applied Wave Mathematics*, 2009.

S. H. Christiansen, A. Gillette. Constructions of some minimal finite element systems. *Mathematical Modelling and Numerical Analysis*, 2016.

B. Cockburn, G. Fu. A systematic construction of finite element commuting exact sequences. arXiv:1605.00132, 2016.

A. Gillette, T. Kloefkorn Trimmed serendipity finite element differential forms. `arXiv:1607.00571`, 2016