Finite Element Exterior Calculus for Evolution Problems

Andrew Gillette

joint work with Michael Holst

Department of Mathematics
University of California, San Diego

http://ccom.ucsd.edu/∼agillette/

Motivation

Poisson's equation: Given f find $u(x)$ such that

$$\begin{cases} 0 = \Delta u + f & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Heat equation: Given f and g, find $u(x,t)$ such that

$$\begin{cases} u_t = \Delta u + f & \text{in } \Omega \subset \mathbb{R}^n, & \text{for } t > 0, \\ u = 0 & \text{on } \partial\Omega, & \text{for } t > 0, \\ u|_{t=0} = g & \text{in } \Omega \end{cases}$$

Finite element exterior calculus (FEEC) provides:

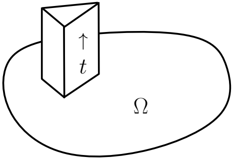
- abstract framework for analyzing numerical approximation of elliptic PDEs
- classification of stable finite element methods with optimal convergence rates

How can the FEEC framework be expanded to classify stable finite element methods for evolutionary PDEs?

Outline of approach

Two possible methods for extending Finite Element Exterior Calculus:

- **Semi-discrete:** Finite element method in space, ODE in time

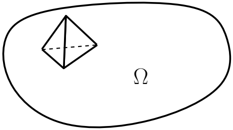


domain: $\Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$

solution basis: $\phi_h|_{t=t_0} : \Omega \to \mathbb{R}$, for each $t_0 \in [0, T]$

error analysis: FEEC + Bochner space theory

- **Fully discrete:** Finite element method in space ***and*** time



domain: $\Omega \times [0, T] \subset \mathbb{R}^n \times \mathbb{R}$

solution basis: $\phi_h : \Omega \times [0, T] \to \mathbb{R}$

error analysis: Finite Element Clifford Algebra *(long term work in progress)*

This talk: Optimal convergence estimates for linear and semi-linear semi-discrete methods for parabolic problems.

Table of Contents

Outline

Finite Element Exterior Calculus: Overview

Consider a mixed method for Poisson's problem on a domain $\Omega \subset \mathbb{R}^n$:

continuous
$$\Delta u + f = 0, \quad u \in H^2$$

mixed weak
$$\begin{aligned} (\text{div}\ \sigma, \phi) + (f, \phi) &= 0, \quad \forall \phi \in H\Lambda^n &= L^2 \\ (\sigma, \omega) + (u, \text{div}\ \omega) &= 0, \quad \forall \omega \in H\Lambda^{n-1} &= H(\text{div}) \end{aligned}$$

mixed FEM
$$\begin{aligned} (\text{div}\ \sigma_h, \phi_h) + (f, \phi_h) &= 0, \quad \forall \phi_h \in \Lambda_h^n &\subset L^2 \\ (\sigma_h, \omega_h) + (u_h, \text{div}\ \omega_h) &= 0, \quad \forall \omega_h \in \Lambda_h^{n-1} &\subset H(\text{div}) \end{aligned}$$

Major Conclusions from FEEC

- The finite elements spaces Λ_h^{n-1} and Λ_h^{n-1} should be chosen from two classes of piecewise polynomial spaces, denoted $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$
- If this choice is made in a compatible manner **implied by the exterior calculus structure**, then optimal *a priori* error estimates are guaranteed

Finite Element Exterior Calculus: Background

FEEC generalizes the standard exact sequence from finite element theory.

The L^2 **deRham complex** for $n = 3\dots$

$$H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2$$

. . .can be viewed as a sequence of Hilbert spaces. . .

$$H\Lambda^0 \xrightarrow{d_0} H\Lambda^1 \xrightarrow{d_1} H\Lambda^2 \xrightarrow{d_2} H\Lambda^3$$

. . .with associated “graph inner products”. . .

$$(u, v)_{H\Lambda^k} := (u, v)_{L^2} + (d_k u, d_k v)_{L^2}$$

. . .and hence we seek solutions in finite-dimensional sub-spaces:

$$\Lambda_h^0 \xrightarrow{d_0} \Lambda_h^1 \xrightarrow{d_1} \Lambda_h^2 \xrightarrow{d_2} \Lambda_h^3$$

Extensions: Arbitrary n; generalized Hodge-Laplacian problems; elasticity problems.

ARNOLD, FALK, WINTHER *Finite Element Exterior Calculus*, Bulletin of the AMS, 2010.

Polynomial Discrete Differential Form Spaces

Finite element spaces Λ_h^k should satisfy three approximation properties:

1. **Subcomplex** of the L^2 deRham complex, i.e. $d\Lambda_h^k \subset \Lambda_h^{k+1}$.
2. Sufficient **approximation power** for upper bounds on $\inf_{v \in \Lambda_h^k} ||u - v||_{H\Lambda^k}$.
3. Bounded **cochain projections** $\pi_h^k : H\Lambda^k \longrightarrow \Lambda_h^k$ that are invariant on Λ_h^k, commute with d, and satisfy $\left|\left|\pi_h^k v\right|\right|_{H\Lambda^k} \leq c\,||v||_{H\Lambda^k}$ for all $v \in H\Lambda^k$.

Two classes of spaces are shown to achieve this:

$\mathcal{P}_r\Lambda^k(\mathcal{T}) :=$ k-forms with coefficients belonging to $\mathcal{P}_r$ on each n-simplex of $\mathcal{T}$.

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) := \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}) \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(\mathcal{T}).$$

where $\mathcal{P}_r$ = polynomials in n variables of degree $\leq r$, $\mathcal{H}$= homogeneous...

$\mathcal{T}$ = simplicial mesh of the domain

κ = Koszul differential map

Examples for $n = 3$:

$$\mathcal{P}_{r+1}\Lambda^2(\mathcal{T}) = \text{Nédélec 2nd-kind } H(\text{div}) \text{ elements of degree} \leq r+1$$

$$\mathcal{P}_{r+1}^-\Lambda^2(\mathcal{T}) = \text{Nédélec 1st-kind } H(\text{div}) \text{ elements of order } r$$

$$\mathcal{P}_{r+1}^-\Lambda^3(\mathcal{T}) = \text{discontinuous elements of degree} \leq r$$

Finite Element Exterior Calculus in context

Theorem [Arnold, Falk, Winther; Bulletin of AMS, 2010]

Assume the elliptic regularity estimate

$$||u||_{H^{s+2}} + ||\nabla u||_{H^{s+1}} + ||\sigma||_{H^{s+1}} + ||\text{div } \sigma||_{H^s} \leq c\,||f||_{H^s}$$

holds for $0 \leq s \leq s_{\max}$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^-\Lambda^n(\mathcal{T}) \;\left(= \mathcal{P}_r\Lambda^n(\mathcal{T})\right)$$

Then for $0 \leq s \leq s_{\max}$, the following error estimates hold

$$||u - u_h||_{L^2} \leq \begin{cases} c h\,||f||_{L^2} & \text{if } r = 0, \\ c h^{2+s}\,||f||_{H^s} & \text{for } r > 0, \quad \text{if } s \leq r - 1 \end{cases}$$

$$||\sigma_h - \sigma||_{L^2} \leq c h^{1+s}\,||f||_{H^s} \quad \text{if} \quad \begin{cases} \Lambda_h^{n-1} = \mathcal{P}_{r+1}\Lambda^{n-1}, & s \leq r + 1, \\ \Lambda_h^{n-1} = \mathcal{P}_{r+1}^-\Lambda^{n-1}, & s \leq r, \end{cases}$$

$$||\text{div } (\sigma_h - \sigma)||_{L^2} \leq c h^{s}\,||f||_{H^s}, \quad \text{if} \quad s \leq r + 1.$$

Semi-discrete Mixed Formulation

Consider a mixed method for the heat equation on $\Omega \subset \mathbb{R}^n$ for $t \in I := [0, T]$.

continuous

$$\begin{aligned} u_t - \Delta u &= f, \\ u|_{t=0} &= g. \end{aligned}$$

mixed weak

$$\begin{aligned} (u_t, \phi) - (\text{div}\ \ \sigma, \phi) &= (f, \phi), && \forall \phi \in H\Lambda^n, && t \in I, \\ (\sigma, \omega) + (u, \text{div}\ \ \omega) &= 0, && \forall \omega \in H\Lambda^{n-1}, && t \in I, \\ u|_{t=0} &= g. \end{aligned}$$

mixed FEM

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div}\ \ \sigma_h, \phi_h) &= (f, \phi_h), && \forall \phi_h \in \Lambda_h^n, && t \in I, \\ (\sigma_h, \omega_h) + (u_h, \text{div}\ \ \omega_h) &= 0, && \forall \omega_h \in \Lambda_h^{n-1}, && t \in I, \\ u_h|_{t=0} &= g_h. \end{aligned}$$

linear system

$$\begin{aligned} AU_t - B\Sigma &= F \\ B^T U + D\Sigma &= 0 && \Rightarrow\ AU_t + BD^{-1}B^T U = F \end{aligned}$$

Semi-discrete Error Bounds

Theorem [Thomée; Galerkin FEM for Parabolic Problems, 1997]

Fix $\mathbf{n} = \mathbf{2}$ and set $\Lambda_h^2 :=$ discontinuous linear, $\Lambda_h^1 :=$ Raviart-Thomas elements.

- If g_h is the discrete solution to the elliptic problem with load data $-\Delta g$:

$$\begin{aligned} ||u_h(t) - u(t)||_{L^2} &\leq ch^2 \left(||u(t)||_{H^2} + \int_0^t ||u_t||_{H^2}\, ds \right), \\ ||\sigma_h(t) - \sigma(t)||_{L^2} &\leq ch^2 \left(||u(t)||_{H^3} + \left(\int_0^t ||u_t||_{H^2}^2\, ds \right)^{1/2} \right). \end{aligned}$$

- Homogeneous case ($f = 0$), g_h as above, $t \geq 0$:

$$||u_h(t) - u(t)||_{L^2} \leq ch^2 |g|_{H^2}, \quad \text{if } g \in \dot{H}^2,$$

$$||\sigma_h(t) - \sigma(t)||_{L^2} \leq ch^3 |g|_{H^3}, \quad \text{if } g \in \dot{H}^3.$$

- Homogeneous case ($f = 0$), $g_h :=$ orthogonal projection of g on to Λ_h^2, $t > 0$:

$$||u_h(t) - u(t)||_{L^2} \leq ch^2 t^{-1} ||g||_{L^2}$$

$$||\sigma_h(t) - \sigma(t)||_{L^2} \leq ch^2 t^{-3/2} ||g||_{L^2}$$

Note: These bounds are ‘space-only’ and restricted to the case $n = 2$.

Bochner spaces and norms

Our new error bounds will employ the theory of **Bochner spaces**

Definition

Let X be a Banach space and $I = (0, T)$. Define

$$C(I, X) := \{u : I \to X \mid u \text{ bounded and continuous}\}$$

Equip this space with the norm

$$||u||_{C(I,X)} := \sup_{t \in I} ||u(t)||_X .$$

The **Bochner space** $L^P(I, X)$ is defined to be the completion of $C(I, X)$ with respect to the norm:

$$||u||_{L^p(I,X)} := \left(\int_I ||u(t)||_X^p \, dt \right)^{1/p} .$$

By setting $X := L^2\Lambda^k(\Omega)$, we get a sequence of **Bochner differential form spaces**:

$$L^2\mathfrak{X}^k := L^2(I, L^2\Lambda^k(\Omega))$$

These are *parametrized* differential form spaces.

Well-Posedness Results with Bochner Spaces

- Let $V \subset H$ be a continuous, dense embedding of real, separable Banach spaces.
  Example: $H^1 \subset L^2$
- Given $A(t) \in \mathcal{L}(V, V^*)$ depending continuously on $t \in I$, define a quadratic form

  $$a(t, u, v) := -(A(t)u, v),$$

  for $(t, u, v) \in \mathbb{R} \times V \times V$.
- Assume that a satisfies the coercivity condition

  $$a(t, u, u) \geq c_1 \, ||u||_V^2 - c_2 \, ||u||_H^2 \, ,$$

  with c_1, c_2 constants independent of $t \in I$.
- Define the **abstract parabolic problem**

  $$\begin{aligned} u_t &= A(t)u + f(t), \\ u(0) &= u_0, \end{aligned}$$

  and the **abstract hyperbolic problem**

  $$\begin{aligned} \ddot{u} &= A(t)u + f(t), \\ u(0) &= u_0, \\ \dot{u}(0) &= u_1. \end{aligned}$$

Well-Posedness Results with Bochner Spaces

Theorem

Given $f \in L^2(I, V^*)$ and $u_0 \in H$, the abstract parabolic problem has a unique solution

$$u \in L^2(I, V) \cap H^1(I, V^*).$$

Example: $H^1 \subset L^2$: The parabolic solution u lies in $L^2(I, H^1) \cap H^1(I, H^{-1})$.

Theorem

Suppose also that

$$a(t, u, v) = a(t, v, u), \quad \forall u, v \in V,$$
$$A \in C^1(I, \mathcal{L}(V, V^*)).$$

Given $f \in L^1(I, H)$, $u_0 \in V$, and $u_1 \in H$, the abstract hyperbolic problem has a unique weak solution

$$u \in C(I, V) \cap C^1(I, H).$$

Example: $H^1 \subset L^2$: The hyperbolic solution u lies in $C(I, H^1) \cap C^1(I, L^2)$.

RENARDY AND ROGERS *An Introduction to PDEs*, Springer-Verlag, 2004.

Outline

Bochner-FEEC Parabolic Error Estimates

parabolic FEM
$$
\begin{aligned}
(u_{h,t}, \phi_h) - (\text{div } \sigma_h, \phi_h) &= (f, \phi_h), \quad \forall \phi_h \in \Lambda_h^n, \quad t \in I, \\
(\sigma_h, \omega_h) + (u_h, \text{div } \omega_h) &= 0, \quad \forall \omega_h \in \Lambda_h^{n-1}, \quad t \in I,
\end{aligned}
$$

Theorem [G, Holst, 2011]

Let $\mathbf{n} \geq \mathbf{2}$ and fix $I = [0, T]$. Suppose the regularity estimate

$$||u(t)||_{H^{s+2}} + ||\nabla u(t)||_{H^{s+1}} + ||\sigma(t)||_{H^{s+1}} + ||\text{div } \sigma(t)||_{H^s} \leq c\, ||f(t)||_{H^s}$$

holds for $0 \leq s \leq s_{\max}$ and $t \in I$. Choose finite element spaces

$$\Lambda_h^{n-1} = \left\{ \begin{array}{c} \mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^{-}\Lambda^{n-1}(\mathcal{T}) \end{array} \right\}, \quad \Lambda_h^n = \mathcal{P}_{r+1}^{-}\Lambda^n(\mathcal{T}) \ \left(= \mathcal{P}_r\Lambda^n(\mathcal{T})\right)$$

Then for $0 \leq s \leq s_{\max}$ and g_h the solution to the elliptic problem we have

$$||u_h - u||_{L^2\mathfrak{X}^n} \leq \left\{ \begin{array}{ll} ch\left(||f||_{L^2(I,L^2)} + \sqrt{T}\,||f_t||_{L^1(I,L^2)}\right) & \text{if } \Lambda_h^n = \mathcal{P}_1^{-}\Lambda^n(\mathcal{T}) \\ ch^{2+s}\left(||f||_{L^2(I,H^s)} + \sqrt{T}\,||f_t||_{L^1(I,H^s)}\right) & \text{for } r > 0, \text{ if } s \leq r-1 \end{array} \right.$$

and. . .

Bochner-FEEC Paraoblic Error Estimates

Theorem [G, Holst, 2011]

$$||\sigma_h - \sigma||_{L^2\mathfrak{X}^{n-1}} \leq \begin{cases} ch\left(||f||_{L^2(I,H^s)} + \sqrt{T}\,||f_t||_{L^2(I,L^2)}\right) \\ \qquad \text{if } r=0,\ s=0,\ \Lambda_h^{n-1} = \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||f||_{L^2(I,H^s)} + h\sqrt{T}\,||f_t||_{L^2(I,L^2)}\right) \\ \qquad \text{if } r=0,\ s\leq 1,\ \Lambda_h^{n-1} = \mathcal{P}_1\Lambda^{n-1}(\mathcal{T}) \\ c\left(h^{1+s}\,||f||_{L^2(I,H^s)} + h^{(3/2)+s}\sqrt{T}\,||f_t||_{L^2(I,H^s)}\right) \\ \qquad \text{for } r>0, \text{ if } s\leq r-1 \end{cases}$$

$$||\text{div}(\sigma_h - \sigma)||_{L^2\mathfrak{X}^{n}} \leq \begin{cases} c\left(h^{s}\,||f||_{L^2(I,H^s)} + h\,||f_t||_{L^2(I,L^2)}\right) \\ \qquad \text{if } r=0,\ s\leq 1 \\ c\left(h^{s}\,||f||_{L^2(I,H^s)} + h^{2+s}\,||f_t||_{L^2(I,H^s)}\right) \\ \qquad \text{for } r>0, \text{ if } s\leq r-1 \end{cases}$$

Proof Sketch

Sketch of proof:

- For any $t_0 \in I$, define the **time-ignorant discrete elliptic problem**:

  Find $(\tilde{u}_h, \tilde{\sigma}_h) \in \Lambda_h^n \times \Lambda_h^{n-1}$ such that

$$\begin{aligned} (\text{div } \tilde{\sigma}_h, \phi_h) + (-\Delta u(t_0), \phi_h) &= 0, \quad && \forall \phi_h \in \Lambda_h^n, \\ (\tilde{\sigma}_h, \omega_h) + (\tilde{u}_h, \text{div } \omega_h) &= 0, \quad && \forall \omega_h \in \Lambda_h^{n-1}, \\ \tilde{u}_h(0) &= g_h. \end{aligned}$$

- Each error term can then be decomposed as:

$$\underbrace{||u(t) - u_h(t)||_{L^2}}_{\text{error between weak and semi-discrete}} \leq \underbrace{||u(t) - \tilde{u}_h(t)||_{L^2}}_{\text{error between weak and time-ignorant elliptic}} + \underbrace{||\tilde{u}_h(t) - u_h(t)||_{L^2}}_{\text{error between time-ignorant elliptic and semi-discrete}}$$

- The first term is bounded by the corresponding Arnold-Falk-Winther estimates.
- The second term is bounded using techniques from Thomée.

Remark: The idea of elliptic projection dates back to papers by Wheeler from the 1970s.

Semi-linear Semi-discrete Parabolic Estimates

Using results from Holst and Stern, we can also analyze semi-linear problems.

Consider a mixed method for the **semi-linear** heat equation on $\Omega \subset \mathbb{R}^n$:

continuous:

$$\begin{aligned} u_t - \Delta u + F(u) &= f, \\ u|_{t=0} &= g. \end{aligned}$$

mixed weak:

$$\begin{aligned} (u_t, \phi) - (\text{div } \sigma, \phi) + (F(u), \phi) &= (f, \phi), & \forall \phi \in H\Lambda^n, & \quad t \in I, \\ (\sigma, \omega) + (u, \text{div } \omega) &= 0, & \forall \omega \in H\Lambda^{n-1}, & \quad t \in I, \\ u|_{t=0} &= g. \end{aligned}$$

semi-discrete:

$$\begin{aligned} (u_{h,t}, \phi_h) - (\text{div } \sigma_h, \phi_h) + (F(u_h), \phi_h) &= (f, \phi_h), & \forall \phi_h \in \Lambda_h^n, & \quad t \in I, \\ (\sigma_h, \omega_h) + (u_h, \text{div } \omega_h) &= 0, & \forall \omega_h \in \Lambda_h^{n-1}, & \quad t \in I, \end{aligned}$$

Semi-linear Error Estimates

Theorem [G, Holst, 2011]

Let $\mathbf{n} \geq \mathbf{2}$. Assume that F is Lipschitz with respect to the L^2 norm, i.e. $\exists\ C$ such that

$$||F(v) - F(w)||_{L^2} \leq C||v - w||_{L^2}, \qquad \forall\ v, w \in L^2$$

Then, under the same assumptions on the regularity of u and choice of finite element spaces, we have **identical estimates to the linear case**.

Sketch of proof:

- Use elliptic projection as before $\longrightarrow$ defines $(\tilde{u}_h, \tilde{\sigma}_h)$.
- Bound $||u(t) - \tilde{u}_h(t)||_{L^2}$ using estimates from
  HOLST, STERN *Semilinear Mixed Problems on Hilbert Complexes and their Numerical Approximation*, Found. Comp. Math., in press, 2010.
- These estimates recover the same rates as the Arnold-Falk-Winther estimates for the linear case.
- The remaining error term is analyzed as in the linear case.

Outline

Significance of the error estimates

$$\left\{ \begin{array}{c} \text{Given bounds on} \\ ||f||_{L^2} \text{ and } ||f_t||_{H^s} \end{array} \right\} + \left\{ \begin{array}{c} \text{Set } \Lambda_h^n, \Lambda_h^{n-1} \\ \text{via FEEC} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Stable method w/} \\ \text{optimal conv. rate} \end{array} \right\}$$

- Results hold in **arbitrary spatial dimension** n, not just $n = 2$.
- **Generic approach** removes need for *ad hoc* choice of finite element spaces.
- Potential applications to the **Yamabe flow** problem, a semilinear parabolic problem used to analyze the **Yamabe problem**: Given a compact Riemannian manifold (M, g) of dimension $n > 3$, find a metric conformal to g with constant scalar curvature.

  BANK, HOLST, SZYPOWSKI, ZHU, *Finite Element Error Estimates for Critical Exponent Semilinear Problems without Angle Conditions*, arXiv:1108.3661, 2011.

- Possible extensions of approach to parabolic vector Hodge-Laplacian problems, hyperbolic problems, and ‘geometric variational crimes’

Geometric Variational Crimes

What is a variational crime?

Variational problem: Find $u \in V$ satisfying

$$B(u, v) = F(v), \quad \forall v \in V.$$

Galerkin variational problem: Find $u_h \in V_h$ satisfying

$$B(u_h, v_h) = F(v_h), \quad \forall v_h \in V_h \subset V$$

Generalized discrete problem: Find $u_h \in V_h$ satisfying

$$B_h(u_h, v_h) = F_h(v_h), \quad \forall v_h \in V_h \not\subset V$$

- Some variational crimes have well-characterized error, e.g. quadrature rules
- Others require customized tools for error analysis e.g. surface finite elements, isoparametric elements, mass lumping, etc.
- The FEEC framework has been extended to analyze such errors in the Hilbert complex framework:

  HOLST, STERN *Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces.*, in review, 2010.

Parabolic (Semi)-Linear Variational Crimes

- The parabolic (semi)-linear variational crime problem is:

$$
\begin{aligned}
(u_{h,t}, \phi_h)_h - (\text{div } \sigma_h, \phi_h)_h + (F_h(u_h), \phi_h)_h &= (f_h, \phi_h)_h, \quad \forall \phi_h \in \Lambda_h^n, \quad t \in I, \\
(\sigma_h, \omega_h)_h + (u_h, \text{div } \omega_h)_h &= 0, \quad \forall \omega_h \in \Lambda_h^{n-1}, \quad t \in I,
\end{aligned}
$$

  where the inner product $(\cdot,\cdot)_h$ is associated to a domain complex $V_h^k \not\subset H\Lambda^k$

- **Example:** The discrete solution is computed over a triangular mesh of a smooth surface (where the true solution lies).
- Using the Holst-Stern framework, we expect to find the standard error estimates for the parabolic problem plus two additional error terms approximating:
  - Error due to $V_h^k \not\subset H\Lambda^k$
  - Error due to approximation of f by f_h and and F by F_h

G, HOLST, *Finite Element Exterior Calculus for Evolution Problems*, in preparation.

Questions?

Thanks for having me back to visit!

- Slides available at http://ccom.ucsd.edu/∼agillette
- Pre-print of paper will be posted soon, or available by request.