Proof of Proposition II.1.3.32

Let (M, d) be a metric space, and assume that it is sequentially compact. Then it is compact, i.e. every open cover has a finite subcover.

The proof in the notes is poorly written. The result is important enough to warrant a careful exposition. Of course there is overlap with the language in the notes.

The argument has four steps. a) M is *totally bounded* (definition below). b) Total boundedness implies that the topology on M has a countable base. c) This implies that every open cover has a countable subcover. d) Finally, every countable open cover has a finite subcover. The assumption of sequential compactness is used in steps a) and d).

a) Claim: for every $m \in \mathbb{Z}_{>0}$, there is a finite set $Z_m = \{z_1, \ldots, z_N\}$, (with N depending on m) such that the balls $B(z_j, 1/m)$ cover M. A metric space with this property is called *totally bounded*.

Suppose this is not true: for some m, no finite number of balls of radius $1/m$ covers M. We show by induction that there is a sequence $\{z_j\}$ with the property that $d(z_i, z_j) \geq 1/m$ for all i, j. The claim now follows, because no subsequence of such a sequence can be Cauchy, and hence there can be no convergent subsequence. This contradicts the sequential compactness hypothesis.

Pick a point z_1. We are assuming that $B(z_1, 1/m)$ does not cover M. Now suppose as induction step that we have found points $z_1, \ldots, z_k$ such that $\cup_{j=1}^{k} B(z_j, 1/m) \subsetneq M$ and $d(z_i, z_j) \geq 1/m$ for all $i, j = 1, \ldots, k$. Choose $z_{k+1} \in M - \cup_{j=1}^{k} B(z_j, 1/m)$. Then since z_{k+1} does not belong to any $B(z_j, 1/m)$, we have $d(z_{k+1}, z_j) > 1/m$ for all $j = 1, \ldots, k$. Furthermore, since we are assuming that no finite number of $1/m$ balls covers M, we must have $\cup_{j=1}^{k+1} B(z_j, 1/m) \subsetneq M$. The sequence with the desired property exists, and the claim is established.

b) Let $Z = \cup_m Z_m$. Let $y_1, y_2, \ldots$ denote the points in Z. Let $\mathcal{B}$ be the collection of balls $B_{nm} := B(y_n, 1/m)$. Claim: this countable collection is a base for the metric topology.

Pick $x \in M$, and an open U_x containing x. There is a ball centered at x that is contained in U_x, say $B(x, \eta)$. Now choose m with $1/m < \eta/2$, and a $y_j \in Z_m$ for which $d(y_j, x) < 1/m$. Then $\mathcal{B} \ni B(y_j, 1/m) \subset B(x, \eta)$. Hence, $\mathcal{B}$ satisfies the first condition for basis. It is easy to show the condition about the intersection of two basis sets, and I leave that to the reader (you).

c) The next step is to show that every open cover has a countable subcover. Let $\{O_\alpha\}$ be an open cover. For every $x \in M$, there is an O_α containing x. Call it O_x. Since $\mathcal{B}$ is a basis, there is a $B(y_j, 1/m)$ containing x and contained in O_x. Call it B_x. Now, $\mathcal{B}' := \{B_x\}$ is an open cover of M, since every x is contained in a set of $\mathcal{B}'$, but since $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{B}$ is countable, the subcollection $\mathcal{B}'$ must also be countable. (There may be uncountably many x and O_x, but there can be only countably many $B' \in \mathcal{B}'$; this just means that a set $B(y_j, 1/m)$ will be the B_x for many different x.) To get the countable subcover, note that each $B \in \mathcal{B}'$ is contained in at least one O_x, by construction. For each of these B, pick one of these O_x and call it O_B. The collection $\{O_B \mid B \in \mathcal{B}'\}$ is a countable subcover.

d) Finally, I show that every countable open cover of M has a finite subcover. Let $\{O_n\}$ be a countable cover that has no finite subcover. Take the first set, O_1, call it O'_1, and pick an $x_1 \in O'_1$. Next, there is a set O_m such that $O_m \subsetneqq O'_1$ (or else $M = O'_1$, contrary to assumption). Write $O'_2 = O_m$. Pick an $x_2 \in O'_2 - O'_1$. Assume for induction that we have $O'_1, \ldots, O'_n$, and for each $k = 2, \ldots, n$ an $x_k \in O'_k - \cup_{j=1}^{k-1} O'_j$. There is an O_m, call it O'_{n+1}, such that $O'_{n+1} - \cup_{j=1}^{n} O'_j \neq \emptyset$. Pick an x_{n+1} in this difference set. These sets O'_n cover M.

In this way, you get a sequence $\{x_n\}$. It is supposed to have a convergent subsequence, $\{x_{n_j}\}$, whose limit x will be contained in one of the sets O'_r. Hence $x_{n_j} \in O'_r$ for all $j >$ some J. But this is a contradiction, since O'_r contains only one element, x_r, of the sequence $\{x_n\}$.

QED, at long last.