Physics Letters A 377 (2013) 1260–1263

Contents lists available at SciVerse ScienceDirect

Physics Letters A

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Wave instability under short-wave amplitude modulations

Benno Rumpf [a,*], Alan C. Newell [b]

[a] Department of Mathematics, Southern Methodist University, Dallas, TX 75275, USA
[b] Department of Mathematics, University of Arizona, Tucson, AZ 85721, USA

ARTICLE INFO

Article history:
Received 21 December 2012
Received in revised form 8 March 2013
Accepted 9 March 2013
Available online 15 March 2013
Communicated by C.R. Doering

Keywords:
Benjamin–Feir instability
Nonlinear waves
Majda–McLaughlin–Tabak equation

ABSTRACT

The instabilities of nonlinear waves with a square-root dispersion $\omega \sim \sqrt{|k|}$ are studied. We present a new type of instability that affects wavelengths of the order of the carrier wave. This instability can initiate the formation of collapses and of narrow pulses.

1. Introduction

Modulational instabilities of monochromatic waves in weakly nonlinear and strongly dispersive systems [1] are not only ubiquitous in nature (in water waves [2], plasma waves [3], optical waves [4], elastic waves [5]), but are responsible for much of the striking behaviors seen in light filamentation [6] and freak waves in fluid mechanics [7] and in optics [8]. This instability causes an increase of initially small amplitude variations of a monochromatic wave, which can ultimately destroy this wave. Such modulations affect a *long* spatial scale that usually scales as one over the small amplitude of the carrier wave. The new finding of this Letter is an instability at a length scale that is *shorter* than the carrier wave.

We investigate this kind of instability for a class of wave equations

$$i\frac{\partial\phi}{\partial t} = L_\alpha\phi + \sigma\phi|\phi|^2 \tag{1}$$

in one dimension that describe the dynamics of a complex amplitude $\phi(x,t)$ in one spatial dimension. The operator L_α is defined via its eigenfunctions and eigenvalues $L_\alpha \exp(ikx) = |k|^\alpha \exp(ikx)$. Eq. (1) can be derived from a Hamiltonian $H = H_2 + H_4$ with $H_2 = \int \omega(k)|a_k|^2\,dk$, $\omega(k) = |k|^\alpha$ and $H_4 = \int \sigma|\phi|^4/2\,dx$, where a_k is the Fourier transform of ϕ. We consider the two important cases $\alpha = 2$ and $\alpha = 1/2$.

$\alpha = 2$, $\omega = k^2$ yields the second order differential operator $L_2 = -\partial^2/\partial x^2$ so that (1) is the nonlinear Schrödinger equation $i\dot{\phi} + \partial^2\phi/\partial x^2 - \sigma\phi|\phi|^2 = 0$. This equation is canonical in the sense that it can be obtained from many wave equations by expanding the linear dispersion about a carrier wavenumber, and by expanding the nonlinearity in powers of the small amplitude. In this context the nonlinear Schrödinger equation describes long modulations of the carrier wave. But here we take the nonlinear Schrödinger equation as an equation of its own right that is satisfied by monochromatic waves $\phi = A\exp(ikx - i(k^2 + \sigma A^2)t)$ where the amplitude A is considered to be real and positive for simplicity. A modulation of this carrier wave is achieved if the amplitude is initially not constant, but varies slightly in space with a wavenumber q as $A + a\cos(qx) + b\sin(qx)$. This leads to a system of coupled modes, namely two at k and at $k \pm q$. The four corresponding frequencies are almost phase matched as $\omega(k+q) + \omega(k-q) - 2\omega(k) = 2q^2$ (Fig. 1(a)). Still, these modes can interact resonantly when the nonlinearity has the appropriate sign to offset the detuning [9]. For $\sigma = -1$ (the focusing nonlinear Schrödinger equation) waves are unstable under long-wave modulations with $q \sim A$.

Majda, McLaughlin and Tabak [10] have studied an equation of the form (1) in which the dispersion relation $\omega = \sqrt{|k|}$ is not analytic and whose graph is concave down (Figs. 1(b)–(d)), in order to mimic the dispersion of gravity waves in deep water. This MMT equation was originally designed to check predictions of wave turbulence theory in one dimension. Numerical investigations show that coherent structures, namely short-wave pulses [11,12] for $\sigma = 1$ and wave collapses [13–16] for $\sigma = -1$ emerge from disintegrating waves in this system.

In order to gain some understanding of these findings, we now consider the instability of monochromatic waves of (1) with $\omega = \sqrt{|k|}$. Fig. 1(b) shows that the four corresponding frequencies at k,

* Corresponding author.
E-mail address: brumpf@mail.smu.edu (B. Rumpf).

0375-9601/$ – see front matter
http://dx.doi.org/10.1016/j.physleta.2013.03.015

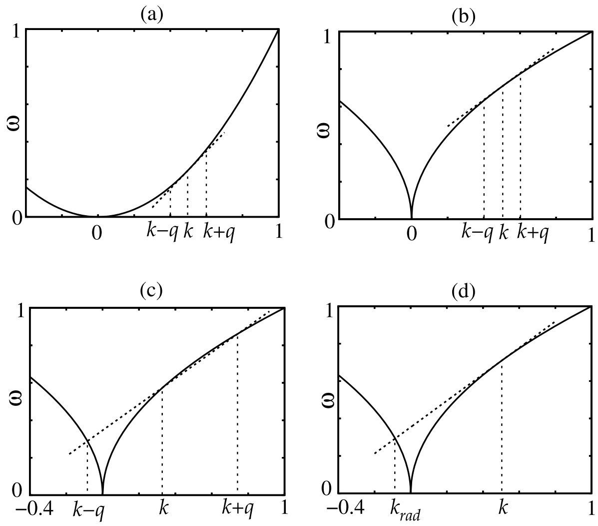

Fig. 1. The dispersion $\omega = k^2$ of the nonlinear Schrödinger equation (a), and the dispersion $\omega = \sqrt{|k|}$ of the MMT equation (b)–(d). The nonlinear Schrödinger equation (a) and the MMT equation (b) admit long-wave modulational instabilities (with $|q| \ll k$) for opposite signs of the nonlinearity. The instability to short-wave amplitude modulations $q \approx 5k/4$ is possible for either sign of the nonlinearity in the MMT equation (c). Solitons in the MMT equation lose power by radiation of resonantly excited modes on the second branch of the dispersion (d).

$k \pm q$ almost match as $\omega(k+q) + \omega(k-q) - 2\omega(k) \approx -q^2/(4k^{3/2})$ for $|q| \ll |k|$. The negative detuning can be offset by a nonlinearity with $\sigma = 1$. This indicates the possibility of a long-wave modulational instability with $q \sim A$. Note that this instability occurs for a positive sign of σ, while the nonlinear Schrödinger equation admits modulational instabilities only for σ negative.

The fact that no such instability is possible for $\alpha = 1/2$, $\sigma = -1$ appears to contradict ample numerical evidence for the existence of wave collapses [13–16] in this equation: How can a wave collapse be initiated without a Benjamin–Feir instability? Fig. 1(c) suggests that the concave down dispersion $\omega \sim \sqrt{|k|}$ admits an additional instability: The resonance condition can be fulfilled for some $q > k > 0$, so that $k - q$ is at the left branch of the dispersion while k and $k + q$ are at the right branch, a case that has not been recognized in the literature on this subject. We investigate this instability in detail and discuss its effects on the formation of coherent structures.

2. Long- and short-wave instabilities

2.1. Linear stability analysis

We perform a stability analysis by perturbing a monochromatic wave with small modulations $|a|, |b| \ll A$ to obtain

$$\phi(x,t) = e^{i(kx-\Omega t)}\big(A + a(t)\cos(qx) + b(t)\sin(qx)\big). \tag{2}$$

The unmodulated wave ($a = b = 0$) has the nonlinear dispersion relation $\Omega = |k|^\alpha + \sigma A^2$. It is assumed with no loss of generality that $k > 0$, $q > 0$. The linearized dynamics of the vector $\mathbf{v} = (\mathrm{Re}(a), \mathrm{Im}(a), \mathrm{Re}(b), \mathrm{Im}(b))^T$ is given by $\dot{\mathbf{v}} = B\mathbf{v}$ with the matrix

$$B = \begin{pmatrix} 0 & D & -M & 0 \\ -D - 2\sigma A^2 & 0 & 0 & -M \\ M & 0 & 0 & D \\ 0 & M & -D - 2\sigma A^2 & 0 \end{pmatrix} \tag{3}$$

and

$$D = \big(|k+q|^\alpha + |k-q|^\alpha - 2|k|^\alpha\big)/2$$

$$M = \big(|k+q|^\alpha - |k-q|^\alpha\big)/2 \geqslant 0. \tag{4}$$

The characteristic equation is

$$\lambda^4 + 2\lambda^2\big(M^2 - Q\big) + \big(M^2 + Q\big)^2 = 0 \tag{5}$$

with $Q = -D(D + 2\sigma A^2)$. The eigenvalues are

$$\lambda_{1,2,3,4} = \pm iM \pm \sqrt{Q}. \tag{6}$$

$Q > 0$ signals an instability $\mathrm{Re}(\lambda) > 0$. Note that the matrix B is non-normal and also defective when the eigenvalues are degenerate either by $M = 0$, or by $D = -2\sigma A^2$ and therefore $Q = 0$, or by $Q = -M^2$. In these cases the eigenvectors don't span the full vector space. This can lead to algebraic growth of perturbations.

It is well known that the focusing nonlinear Schrödinger equation ($\sigma = -1$) is modulationally unstable with $\mathrm{Re}(\lambda) = \pm\sqrt{q^2(2A^2 - q^2)}$ for $2A^2 > q^2$, whereas waves of the defocussing ($\sigma = 1$) nonlinear Schrödinger equation are not modulationally unstable. To study the square-root dispersion $\alpha = 1/2$ we introduce the parameter $s = q/k$ and the rescaled function

$$\begin{aligned} R(s) &= D/|k|^{1/2} \\ &= \big(|1+s|^{1/2} + |1-s|^{1/2} - 2\big)/2. \end{aligned} \tag{7}$$

This function is depicted in Fig. 2. It has zeros $R(s = 0) = R(s = 5/4) = 0$, and $R(s) < 0$ for $0 < s < 5/4$ with a cusp-shaped minimum at $s = 1$, $R(s = 1) = 1/\sqrt{2} - 1$. $R(s)$ is positive for $s > 5/4$ with $R(s) \sim \sqrt{s}$ for large s.

2.2. Domains of the instability

2.2.1. Zero eigenvalues and maximum eigenvalues

To determine the domain of instability we first consider its boundaries $\mathrm{Re}(\lambda) = \sqrt{Q} = 0$. This is satisfied by $D = 0$ and for $D = -2\sigma A^2$. $D = 0$ or equivalently

$$R(s) = 0 \tag{8}$$

is solved by $s = 0$ and $s = 5/4$ and yields the two lines $q = 0$ and $q = 5k/4$ where $\mathrm{Re}(\lambda) = 0$. The condition $D = -2\sigma A^2$ can be written as

$$R(s) = -2\sigma|\tilde{k}|^{-1/2} \tag{9}$$

with the rescaled dimensionless wavenumber $\tilde{k} = k/A^4$ (k and A^4 have the same dimensions).

To determine the maximum of $\mathrm{Re}(\lambda)$ we note that $\partial Q/\partial k = 0$ and $\partial Q/\partial q = 0$ are both satisfied at $D = -\sigma A^2$. In rescaled variables this reads

$$R(s) = -\sigma|\tilde{k}|^{-1/2}. \tag{10}$$

We discuss the domains with $\mathrm{Re}(\lambda) > 0$ separately for $\sigma = 1$ and $\sigma = -1$.

2.2.2. Instabilities for $\sigma = 1$

Condition (9) corresponds to intersection points of the lines $R(s)$ and $-2|\tilde{k}|^{-1/2}$ in Fig. 2. There are two such intersection points at s_1 and s_2 if the minimum of $R(s)$ is below the line $-2|\tilde{k}|^{-1/2}$. In the intervals $0 < s < s_1$ and $s_2 < s < 5/4$ we obtain $Q > 0$ and therefore $\mathrm{Re}(\lambda) > 0$.

For high values of the amplitude $A^4 > (\sqrt{2} - 1)^2|k|/8$ there is no intersection of $R(s)$ and the line $-2|\tilde{k}|^{-1/2}$. In this case all modulations with $0 < s < 5/4$ (that is $0 < q < 5k/4$) yield $\mathrm{Re}(\lambda) > 0$.

We can express k and q at $\mathrm{Re}(\lambda) = 0$ as parametric functions of s. With the scaled wavenumbers $\tilde{q} = q/A^4$ and $\tilde{k} = k/A^4$ the parametric curves with $\mathrm{Re}(\lambda) = 0$ are given by

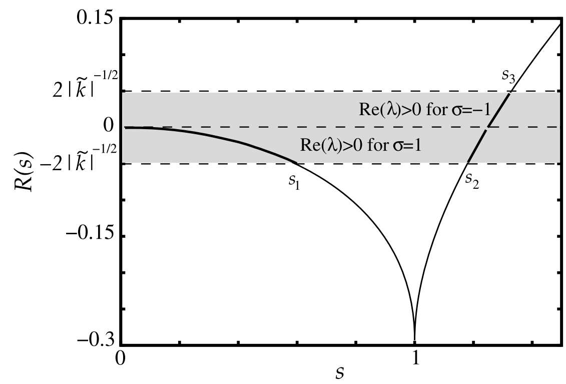

Fig. 2. The function $R(s) = (|1+s|^{1/2} + |1-s|^{1/2} - 2)/2$ with $s = q/k$. $R(s) = 0$ at $s = 0$ and $s = 5/4$ and $R(s) = -2\sigma|\tilde{k}|^{-1/2} = -2\sigma A^2|k|^{-1/2}$ at s_1, s_2, s_3 correspond to eigenvalues λ with a zero real part. Eigenvalues with a positive real part are obtained in the case $\sigma = 1$ for $0 < s < s_1$ and $s_2 < s < 5/4$, and in the case $\sigma = -1$ for $5/4 < s < s_3$.

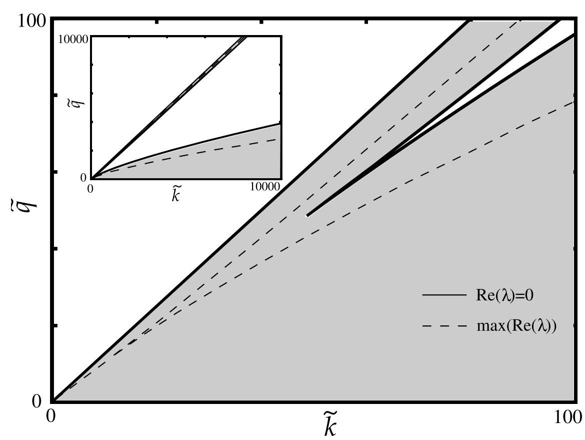

Fig. 3. Instabilities for $\sigma = 1$ depending on $\tilde{k} = k/A^4$, $\tilde{q} = q/A^4$. In the shaded regions there is an eigenvalue with $\mathrm{Re}(\lambda) > 0$ while the eigenvalues are imaginary outside these regions. The dotted curves indicate local maxima of $\mathrm{Re}(\lambda)$ with respect to $\tilde{q}$. Insert: Instabilities for large $\tilde{k}$, $\tilde{q}$, which corresponds to small A. There is a long-wave modulational instability with $\tilde{q} \sim \tilde{k}^{3/4}$ or $q \sim Ak^{3/4}$. In addition, there is an instability against short-wave modulations in the narrow channel at below $q = 5k/4$.

$$\tilde{k}(s) = 4/R^2(s)$$
$$\tilde{q}(s) = 4s/R^2(s). \tag{11}$$

The condition (10) for maxima of $\mathrm{Re}(\lambda) = \sqrt{Q}$ with respect to q and k yields the parametric curve

$$\tilde{k}(s) = 1/R^2(s)$$
$$\tilde{q}(s) = s/\big(R^2(s)\big). \tag{12}$$

Eq. (10) cannot be satisfied if $-|\tilde{k}|^{-1/2} < R(s=1) = 1/\sqrt{2} - 1$, that is the line $-|\tilde{k}|^{-1/2}$ is below the minimum of $R(s)$. In this case, the highest possible value of $\mathrm{Re}(\lambda)$ is achieved at $s = 1$, that is at $\tilde{k} = \tilde{q}$ or $k = q$.

Fig. 3 shows the curves $\mathrm{Re}(\lambda) = 0$ in a $\tilde{k}$–$\tilde{q}$ diagram with $0 \leqslant \{\tilde{k}, \tilde{q}\} \leqslant 100$ and with $0 \leqslant \{\tilde{k}, \tilde{q}\} \leqslant 10\,000$ (insert). High values of $\tilde{k}, \tilde{q} \sim A^{-4}$ correspond to a small amplitude A. Positive real parts of the eigenvalues are achieved in the shaded regions. This region of instability lies below the line $\tilde{q} = 5\tilde{k}/4$, and it is split in two parts by a wedge-shaped region where the eigenvalues are imaginary. The tip of this wedge is given by $-2|\tilde{k}|^{-1/2} = R(1)$, $\tilde{k} = \tilde{q} = 8/(3 - 2\sqrt{2})$.

In the limit of small amplitudes $A \ll 1$ the intersection points in Fig. 2 are at $s_1 \ll 1$ and $s_2 \approx 5/4$. With $R(s_1) \approx -s_1^2/8$ Eq. (9) yields $\tilde{q} = 4\tilde{k}^{3/4}$ or $q = 4Ak^{3/4}$. This is the lower boundary of the wedge-shaped region $\mathrm{Re}(\lambda) = 0$. The intersection point s_2 yields the upper boundary of the wedge as $\tilde{q} = 5\tilde{k}/4 - 3\tilde{k}^{1/2}$ or $q = 5k/4 - 3A^2k^{1/2}$. Between the upper boundary of the wedge and the line $\tilde{q} = 5\tilde{k}/4$ there is a narrow channel with $\mathrm{Re}(\lambda) > 0$. This signifies an instability against modulations with a wavelength that is *shorter* than the wavelength of the carrier wave, corresponding to $k < q < 5k/4$. To conclude, waves are affected both by long-wave ($q \sim A$) and by short-wave ($q \approx 5k/4$) instabilities.

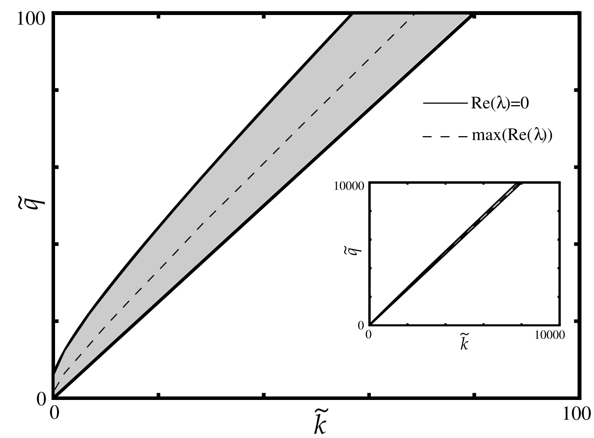

Fig. 4. Instabilities for $\sigma = -1$ depending on $\tilde{k}$ and $\tilde{q}$: The shaded narrow corridor above the curve $q = 5k/4$ indicates eigenvalues $\mathrm{Re}(\lambda) > 0$. The dotted curve indicates a local maximum of $\mathrm{Re}(\lambda)$ with respect to $\tilde{q}$. This is an instability against short-wave modulations with $q > 5k/4$. The homogeneous mode is unstable under modulations with $q \sim A^4$. There is no instability against long-wave modulations $q < k$.

2.2.3. Instabilities for $\sigma = -1$

$R(s)$ is unbounded from above, so for $\sigma = -1$ it is always possible to satisfy condition (9), $R(s) = 2\tilde{k}^{-1/2}$. The intersection point $s_3 > 5/4$ corresponds to $\mathrm{Re}(\lambda) = 0$.

In the limit of small A, $s_3 \approx 5/4$, we obtain the curve $\tilde{q} = 5\tilde{k}/4 + 3\tilde{k}^{1/2}$ or $q = 5k/4 + 3A^2k^{1/2}$.

Fig. 4 depicts a shaded region with $\mathrm{Re}(\lambda) > 0$ and its boundaries $\mathrm{Re}(\lambda) = 0$. This is a narrow band above the line $q = 5k/4$. At $k = 0$ all modes with $q < 4A^4$ are unstable. There is no instability region below the line $\tilde{q} = 5\tilde{k}/4$. To conclude, waves with $k \gg A^4$ are unstable only under short-wave modulations, that is modulations that are shorter than the carrier wave.

3. Discussion

3.1. Long-wave and short-wave instability for $\sigma = 1$: Emergence of radiating pulses

In [11] we studied a damped and driven version of the MMT equation and found spontaneously forming traveling pulses. These pulses are similar to bright solitons, but they contain essentially only one single loop. The formation of such pulses originates from the instability of waves under short wavelength modulations with $|q| \approx 5|k|/4$ (Fig. 1(c)). This causes amplitude variations at a length scale that is shorter than the scale of phase variations. Eventually the amplitude becomes close to zero locally. As a result the wave disintegrates into short pulse-like structures that vary in phase and amplitude at similar scales $\sim 2\pi/k$ and $\sim 2\pi/q$.

A second finding in [11] is that these traveling pulses lose power by radiating low-amplitude waves. The reason for this is that the pulse resonantly excites linear waves, a mechanism that is known from quasisolitons [15]. As it turns out, this emission of radiation is possible only if the tangent of the dispersion curve at a point k intersects the opposite branch of the dispersion relation at a point k_{rad} (with $k_{rad} = -(\sqrt{2} - 1)^2 k$ for the square root dispersion, see Fig. 1(d)). Three of the four interacting modes are near k and contribute to the pulse, and one mode k_{rad} for the radiation is on the opposite branch of the dispersion. This distribution of modes has some similarity with that of Fig. 1(c) where three modes at the right branch (two at k and one at $k+q$) interact with one at the left branch (at $k-q \approx -k/4$). It is remarkable that both the modulational instability (which creates the pulse) and the radiation (which leads to the decay of the pulse) rely on interactions between modes on both branches of the dispersion.

3.2. Short-wave instability for $\sigma = -1$: Emergence of collapses

Numerical studies [13,14,16] of the MMT equation with $\sigma = -1$ have encountered wave collapses, that is the formation of high-amplitude structures in finite time. The collapses lead to a surge in the two energy contributions H_2 and H_4 while the total energy is conserved since the quadratic part H_2 and the quartic part H_4 of the Hamiltonian have opposite signs. Again, we have to attribute the appearance of these strongly nonlinear structures to the instability by short modulations $q \approx 5k/4$ for $k \gg A^4$. The homogeneous mode is unstable under modulations $q \sim A^4$, but there is no conventional Benjamin–Feir instability in this case.

In the numerical experiments of [16] an external force supplies wave action and energy to the system by exciting long waves. Damping is applied at high wavenumbers (to absorb energy from a direct cascade), and at very small wavenumbers (to absorb wave action from an inverse cascade; this damping affects waves that have a bigger wavelength than the waves that are excited by the driving force). The driven modes have the highest amplitudes and are most unstable. This instability then initiates the finite-time blow-ups of the amplitude.

Surprisingly, wave collapses emerge only when the strength of the external force exceeds a certain threshold [16]. The system then achieves a statistically stationary nonequilibrium state in which wave turbulence coexists with strongly nonlinear collapses. In contrast to this, wave turbulence without high-amplitude events is observed when a driving force below this strength is applied.

Fig. 4 suggests an answer to the question why no collapses appear for small driving forces. In this case the average amplitudes are small, so possible instabilities concern high values of $\tilde{k}$ and $\tilde{q}$. Consequently only a narrow band of modulations destabilizes a mode with k. Now with $q \approx 5k/4$, the mode $k-q \approx -k/4$ is in the low-$|k|$ damping range. The damping diminishes the amplitude of this mode and suppresses the instability.

On the other side, a strong external driving force leads to higher amplitudes, and therefore to smaller values of $\tilde{k}$, $\tilde{q}$. In this case the band of the modulational instability is broader (Fig. 4). Correspondingly the range of possible values of $k-q$ that leads to an instability is also broader. If $k-q$ is outside the low-wavenumber damping range, this modulation is not affected by the damping. For this reason the modulational instability is expected to take place if the amplitude is sufficiently high. We will study this behavior in detail in a longer paper.

3.3. Defocussing and focusing behavior

It appears to be problematic to refer to Eq. (1) with $\alpha = 1/2$, $\sigma = \pm 1$ as "defocusing" and "focusing" in analogy with the nonlinear Schrödinger equation with $\sigma = \pm 1$ [13]: For the "defocusing" MMT equation with $\alpha = 1/2$, $\sigma = 1$ there is a modulational instability. One can derive a *focusing* nonlinear Schrödinger equation to describe modulations of a carrier wave in this equation. This is related to the fact that the "defocusing" MMT equation supports bright quasisolitons [11,15]. However, collapses are not possible because H_2 and H_4 are both positive. For the "focusing" MMT equation with $\sigma = -1$, the signs of H_2 and H_4 are opposite so that collapses are possible, but waves (with the exception of the homogeneous mode) are not unstable against long-wave modulations: Modulations of a carrier wave are described by a *defocussing* nonlinear Schrödinger equation. This is due to the concave down dispersion $d^2\omega/dk^2 < 0$ with $\omega > 0$. In contrast to this, for the nonlinear Schrödinger equation with $d^2\omega/dk^2 > 0$, $\omega > 0$ the possibility of collapses (in two or more dimensions) coincides with the modulational instability. It seems to us that avoiding the terms "focusing/defocusing" in the context of the MMT equation could prevent misunderstandings.

There is no simple way to predict the behavior that results from modulational instabilities. Irreversible processes such as collapses or radiating pulses as in the MMT equation are not always obtained in nonintegrable systems. An example somewhat similar to the system studied here are Rossby waves where periodic behavior results from a modulational instability when the amplitude of the carrier wave is not too high [17]. The same kind of behavior was observed in the discrete nonlinear Schrödinger equation [18], where one can suspect the breakup of KAM-tori as the cause for irreversibility at higher amplitudes.

4. Conclusions

Nonlinear wave dynamics with a concave down dispersion function such as $\omega = \sqrt{|k|}$ can exhibit a modulational instability where the wavelength of the modulation is smaller than the wavelength of a carrier wave with $k \neq 0$. We expect this type of instability to be widespread in systems with four-wave interaction: Its occurrence depends on a global feature of the dispersion relation, namely the existence of three equidistant intersection points of a straight line and the dispersion curve (Fig. 2(c)). It affects modulations with a wavelength of the order of the carrier wave. This short-wave instability can co-occur with a normal long-wave Benjamin–Feir instability (as for $\sigma = 1$), but it can also affect systems with no such instability (for $\sigma = -1$).

The short-wave instability can have dramatic effects on the dynamics as it enhances short pulses. This is the precursor of the important fully nonlinear events that have been observed numerically in previous papers, namely short radiating pulses (for $\sigma = 1$), and wave collapses (for $\sigma = -1$). Such events have been shown to be crucial for explaining anomalous turbulent behavior in this system [11].

References

[1] V.E. Zakharov, L.A. Ostrovsky, Physica D 238 (2009) 540.
[2] T.B. Benjamin, J.E. Feir, J. Fluid Mech. 27 (1967) 417.
[3] V.I. Karpman, E.M. Krushkal', Zh. Eksperim. Teor. Fiz. 55 (1968) 530, Soviet Phys. JETP 28 (1969) 277.
[4] R.Y. Ciao, E. Garmire, C.H. Townes, Phys. Rev. Lett. 13 (1964) 479.
[5] G. During, C. Josserand, S. Rica, Phys. Rev. Lett. 97 (2006) 025503.
[6] E. Garmire, R.Y. Ciao, C.H. Townes, Phys. Rev. Lett. 16 (1966) 347.
[7] A.I. Dyachenko, V.E. Zakharov, Pisma Zh. Eksperim. Teor. Fiz. 88 (2008) 356, JETP Lett. 88 (2008) 307.
[8] D.R. Solli, C. Ropers, P. Koonath, B. Jalali, Nature 450 (2007) 1054.
[9] V.E. Zakharov, Zh. Eksperim. Teor. Fiz. 51 (1966) 1107, Soviet Phys. JETP 24 (1967) 740.
[10] A.J. Majda, D.W. McLaughlin, E.G. Tabak, J. Nonlinear Sci. 7 (1997) 9.
[11] B. Rumpf, A.C. Newell, V.E. Zakharov, Phys. Rev. Lett. 103 (2009) 074502.
[12] A.C. Newell, B. Rumpf, V.E. Zakharov, Phys. Rev. Lett. 108 (2012) 194502.
[13] D. Cai, A.J. Majda, D.W. McLaughlin, E.G. Tabak, Proc. Natl. Acad. Sci. 96 (1999) 14216;
D. Cai, A.J. Majda, D.W. McLaughlin, E.G. Tabak, Physica D 152–153 (2001) 551.
[14] V.E. Zakharov, P. Guyenne, A.N. Pushkarev, F. Dias, Physica D 152–153 (2001) 573;
V.E. Zakharov, P. Guyenne, A.N. Pushkarev, F. Dias, Contemp. Math. 283 (2001) 107.
[15] V. Zakharov, F. Dias, A. Pushkarev, Phys. Rep. 398 (2004) 1.
[16] B. Rumpf, L. Biven, Physica D 204 (2005) 188.
[17] C.P. Connaughton, B.T. Nadiga, S.V. Nazarenko, B.E. Quinn, J. Fluid Mech. 654 (2010) 207.
[18] B. Rumpf, Physica D 238 (2009) 2067.