# Introduction to piecewise flat manifolds

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## 1 Manifolds and triangulations in 2D

Recall that a manifold is a space which is locally homeomorphic to $\mathbb{R}^n$, that is, if $M$ is a manifold, then for every point $x \in M$, there is an open set $U \subset M$ containing $x$ (a "neighborhood of $x$") such that $U$ is homeomorphic to $\mathbb{R}^n$. Note the following consequence of this definition:

- Note that since $\mathbb{R}^n$ is NOT homeomorphic to $\mathbb{R}^m$ if $m \neq n$ (this is a difficult fact called invariance of domain), every point has a neighborhood homeomorphic to $\mathbb{R}^n$ with the same $n$ for every point. This value of $n$ is called the *dimension* of $M$. $M$ is then called an $n$-manifold, and often denoted $M^n$.

How do you specify a manifold? We will be interested in a triangulated manifold.

**Definition 1** *A triangulation of a 2-manifold is a collection of (abstract) triangles glued together along their boundaries, satisfying certain properties, such that the resulting object satisfies the manifold property.*

**Example 2** *Not a manifold: three triangles glued along an edge.*

**Example 3** *Manifolds: boundary of a tetrahedron, icosahedron, octahedron, double triangle, torus, many holed torus*

Note that a triangulation has the following pieces as part of it:

- A collection of abstract points called vertices.
- A collection of abstract line segments called edges.
- A collection of abstract triangles called faces. The boundary of the triangle should consist of three (not necessarily distinct) edges.

To be a triangulation, we require the following:

- The ends of an edge should correspond to (not necessarily distinct) vertices.

- The boundary of a face should consist of three (not necessarily distinct) edges.

What makes a triangulation into a manifold? It needs to satisfy the following two properties:

- Each edge neighbors exactly two faces.
- At each vertex, the neighboring faces can be arranged in an order $f_1, f_2, f_3, \ldots, f_k = f_1$ such that $f_j$ and $f_{j+1}$ intersect at an edge which has the vertex on the boundary.

## 2 Piecewise flat surfaces

A triangulated manifold is given a piecewise flat structure by specifying lengths of edges in such a way that triangles can be formed from those edge lengths. This supposed the following:

**Proposition 4** *Given three positive numbers $\ell_1, \ell_2, \ell_3$ such that*

$$\ell_i + \ell_j > \ell_k$$

*for $\{i, j, k\} = \{1, 2, 3\}$, there is a unique (Euclidean) triangle (up to rigid motion in the plane) with those side lengths.*

**Proof.** Place $\ell_1$ along the positive $x$-axis. Attach a line of length $\ell_2$ at its second vertex. By rotating the edge, we can get any distance for the third side between $\ell_1 - \ell_2$ and $\ell_1 + \ell_2$. We have assumed that $\ell_3$ is in that range. There are two possible triangles, but they are the same up to rigid motion. ■

So, the edge lengths determine the geometry of each of the triangles. Note also that the edge lengths determine angles and areas:

**Proposition 5** *The area $A$ of a triangle can be computed directly from its edge lengths. The angles of a triangle can also be computed from its edge lengths.*

**Proof.** The formula for the area of a triangle in terms of edge lengths is called Heron's formula, and is

$$A(\ell_1, \ell_2, \ell_3) = \frac{1}{2}\sqrt{(\ell_1 + \ell_2 + \ell_3)(-\ell_1 + \ell_2 + \ell_3)(\ell_1 - \ell_2 + \ell_3)(\ell_1 + \ell_2 - \ell_3)}.$$

It can also be expressed in terms of the Cayley-Menger determinant:

$$A = \frac{1}{4}\sqrt{-\det\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & \ell_1 & \ell_2 \\ 1 & \ell_1 & 0 & \ell_3 \\ 1 & \ell_2 & \ell_3 & 0 \end{pmatrix}}.$$

Angles can be computed using the law of cosines, which says

$$\cos\gamma_1 = \frac{\ell_2^2+\ell_3^2-\ell_1^2}{2\ell_1\ell_2}$$

if $\gamma_1$ is the angle opposite the side of length $\ell_1$. ■

Finally, we introduce curvature.

**Definition 6** *The curvature at the vertex $v$ of a piecewise flat surface is*

$$K_v = 2\pi - \sum_{f>v} \gamma_{v<f}$$

*where the sum is over all faces containing $v$.*

Notice that this is zero precisely if the "flower" of the vertex can be drawn flat in the plane.

It will be useful to have the following invariant of a surface:

**Definition 7** *The Euler characteristic of a surface $M$ is defined to be*

$$\chi(M) = |V| - |E| + |F|,$$

*where $|V|$ is the number of vertices, $|E|$ is the number of edges, and $|F|$ is the number of faces.*

The important fact is the following:

**Theorem 8** *The Euler characteristic depends only on the topology of the manifold, not the triangulation.*

So, in particular, any regular polytope has the same Euler characteristic, that of the two-sphere.

The following relates the Euler characteristic to the curvature:

**Theorem 9** *(Discrete Gauss-Bonnet)*

$$\sum_v K_v = 2\pi\chi.$$

# 3 Three-manifolds

This can be adapted to higher dimensional manifolds, in particular three-dimensions. How do we adapt the surface definition to three-manifolds?

- A triangulation consists of tetrahedra glued together along their boundary.
- To be a manifold, one must have:
  - Every face (triangle) borders exactly 2 tetrahedra.

  - Around every edge, we can order the tetrahedra bordering it as $t_1, t_2, \ldots, t_k = t_1$ such that $t_j$ and $t_{j+1}$ intersect at a face bordering the edge.
  - At every vertex the Euler characteristic of the link is 2 (that of the sphere). [The link is the sperical polyhedron determined by the trihedra.]

- Volume can be calculated from edge lengths by Cayley-Menger determinant

- Dihedral angles of a tetrahedron can be computed using spherical law of cosines:

$$\cos \beta_{ij,k\ell} = \frac{\cos \gamma_{i,k\ell} - \cos \gamma_{i,jk} \cos \gamma_{i,j\ell}}{\sin \gamma_{i,jk} \sin \gamma_{i,j\ell}}$$

- Curvature at edges is

$$K_e = \left( 2\pi - \sum_{t>e} \beta_{e<t} \right) \ell_e.$$