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Computer Algebra Systems in General Relativity

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Abstract

This paper presents a review of the use of Computer Algebra Systems in General Relativity research and teaching. On one hand, the impact of using Computer Algebra Systems in General Relativity research is illustrated by pointing out some important achievements in the field. In particular, by using Computer Algebra Systems, the present author has been able to obtain results that would have been almost impossible otherwise. On the other hand, Computer Algebra Systems can be a very helpful tool in teaching and learning GR. Some reports on using Computer Algebra Systems in teaching General Relativity are outlined.

1. Introduction

General Relativity (GR) is Einstein's theory of gravitation in which spacetime is a 4-dimensional Riemannian manifold. The spacetime geometry is described by the metric tensor, and the energy distribution is described by the energy tensor. The interaction between the spacetime and the energy is governed by Einstein's field equations and the equations of motion, in such a way that the spacetime geometry is determined by the energy tensor, and the motion of matter is determined by the metric tensor. According to Wheeler: "matter tells space how to curve, and space tells matter how to move." The field equations are given in tensorial form and, in general, constitute a nonlinear system of partial differential equations. Hence, the study of GR involves a large number of problems requiring very tedious, time-consuming, and error-prone algebraic manipulations. For this reason, GR was one of the earliest fields of application of Computer Algebra Systems (CAS).

Programs for GR need special features for tensor algebra, both in components and in indicial form. Thus, besides using general-purpose systems, many specialized systems for GR calculations have been developed. The special-purpose systems SHEEP and OrtoCartan are two of the most frequently used systems for field equations in GR. On the other hand, special packages within general-purpose systems have been developed for such special purposes as indicial calculations, tensor component calculus and differential forms. Reviews of special-purpose systems and packages are given in [1-5].

This paper presents a review of the use of CAS in GR research and teaching. In section 2 the impact of using CAS in GR research is illustrated by pointing out some significant achievements in the field. In particular, by using CAS, the present author has been able to obtain results that would have been almost impossible otherwise. On the other hand, CAS can be a very helpful tool in teaching and learning GR. In section 3 some projects for using CAS in teaching GR are described.

2. CAS in GR Research

CAS have perhaps most frequently been used in GR for exact solutions in four directions: checking existing solutions, giving unique characterizations, developing online databases of solutions, and finding new solutions.

An early example on checking existing solutions is the work of d'Inverno and Russell-Clark [6] where they used LAM (Lisp Algebraic Manipulator) for classifying 40 metrics originally obtained using indirect methods by Harrison [7]. It has been shown that 4 of them were not in fact vacuum solutions.

More recently, Delgaty and Lake [8] used GRTensorII [9], a special package that runs under Maple, for checking the physical acceptability of isolated static spherically symmetric perfect fluid solutions of Einstein's equations. They tested 127 solutions for a set of regularity and energy conditions. Only 16 of these solutions have been found to satisfy all criteria.

An important contribution of CAS to exact solutions is the study of inhomogeneous cosmologies by Krasiniski [10], where many of the solutions considered were processed by Ortocartan [11], a specialized CAS written in LISP. Besides calculating standard tensors associated with a given metric forms, Ortocartan has additional subprograms for various purposes.

The present author used Mathematica to obtain exact solutions. For the case of a static and spherically symmetric perfect fluid, a class of nine 3-parameter solutions has been obtained [12]. Each solution has been shown to be physically reasonable in some range of the parameters over some region of spacetime, and thus could represent a relativistic stellar model. The class includes Tolman solutions I and IV besides seven new solutions. For the case of a static and cylindrically symmetric perfect fluid, a new class of two 1-parameter solutions and two 2-parameter solutions has been obtained [13] using a special ansatz. One of the 2-parameter solutions includes two previously known solutions as special cases. Using a more general ansatz, another class of six new 3-parameter solutions has been obtained [14]. Under a specific set of conditions, each solution of the two classes could be a source for the vacuum Levi-Civita metric. The use of CAS has been essential for obtaining each of the classes. In particular, the derivation of the latter class involved so complicated symbolic calculations that it would have been almost impossible to be completed without using CAS.

An important problem related to exact solutions is the equivalence problem [15]: given two metrics, does there exist a local transformation between them? The problem has been posed by the repeating discovery of solutions in the literature where many solutions have been initially announced as new, but turned out later to be equivalent to already known solutions in different coordinate systems. Cartan showed that the answer to the equivalence problem depends on computing the tenth covariant derivative of the Riemann tensor of each metric. Karlhede [16] suggested an approach using invariant classification of metrics: if two metrics have the same classification then they are candidates for equivalence, otherwise they are necessarily inequivalent. The problem then reduces to solving four algebraic equations. The order of the covariant derivative needed reduces from the original 10 of Cartan to the range from 2 to 7, depending on the Petrov type of the Riemann tensor. Karlhede's algorithm has been implemented in an extension of SHEEP [17] called CLASSI [18]. Indeed, CLASSI has been used to show that three of the Harrison metrics [6,7] are in fact equivalent.

The advent of the classification algorithm and the rapid development of the Internet have led to the setting up of databases of exact solutions on the Internet. The first such database was developed by Skea [19-21], primarily as an invariant classification database based on CLASSI. The database provides access to a unique characterization of the Riemann tensor and its covariant derivatives for over 200 metrics. In addition, other discrete invariant information is available, such as the Petrov type, Segre type, isotropy group and dimension of isometry group. Unfortunately this database has not been modified since 1998.

Skea's database is static in the sense that the information that can be retrieved is resident in the records. Ishak and Lake [22] described an interactive database GRDB [23] of geometric objects in differential geometry. Database objects include, but are not restricted to, exact solutions of Einstein's field equations. GRDB is non-static in the sense that the information that can be retrieved is not restricted to information resident in the records. This development has become feasible using several platform-independent Java technologies (e.g. Applets, Servlets, JDBC) and following a fully modular object-oriented design. GRDB uses GRTensorII for performing online tensor calculations under Maple. The dynamic nature of the database is accomplished via the inclusion of GRTensorJ [24], an interactive programmable graphical user interface to GRTensorII. The highly interactive nature of GRDB allows systematic internal self-checking and minimization of the required internal records. GRDB has not been updated since 2002.

Another problem related to exact solutions is the determination of hidden symmetries. Jerie *et al* [25] used Dimsym [26], a package for Reduce [27], to find a wide variety of spacetime symmetries, including isometry groups, conformal motions and projective collineations. Besides, Dimsym can be used to find Lie symmetries of the geodesic equations. The package works by forming and solving the determining equations for the generators of symmetries of differential equations.

A third problem related to exact solutions is that of matching two solutions across a boundary surface, where boundary conditions, called junction conditions, should apply. There are various sets of junction conditions, all involving lengthy calculations. GRjunction [28] is a package developed for the evaluation of junction conditions and the parameters associated with thin shells. The package runs under Maple in conjunction with GRTensorII. Besides using the package to verify some examples from the literature, it has been used to join two Kerr solutions with differing masses and angular momenta at a spacelike shell in the slow rotation limit [29]. Also, two Kerr-Newman solutions have been joined at a non-horizon straddling null shell and at a horizon straddling shell [30].

CAS have also been used in Numerical Relativity. There are many important physical situations such as 2-body systems, radiative sources, and interiors of rotating sources, where no exact solutions are known. The study of such problems is only possible by solving Einstein's equations numerically. Numerical Relativity often involves long and complicated algebraic computations, which can be first generated by a CAS. In his pioneering work, Nakamura [31] used Reduce to first generate such large algebraic expressions and then exploited Reduce's ability to convert algebraic expressions into their Fortran equivalent, prior to numerical computation. Numerical Relativity has been widely using CAS since then.

Numerical Relativity is also concerned with numerical investigations of exact solutions such as geodesic tracing. GRworkbench [32-33] is a 3-D visual software tool for such numerical investigation. It has been used to investigate the geodesic paths in the Schwarzschild spacetime [32] and in the spacetime of a Kerr black hole [34]. A more complex problem is the modeling in GRworkbench of idealized interferometers orbiting the center of our galaxy [33]. This model was used to investigate a claim [35] that the mass of our galaxy can be measured using a small 10 cm Michelson-type interferometer located on the surface of the Earth. It has been reported [36] that the claim was found to be an artifact of the simplifying assumptions employed in the analysis. Using GRworkbench, a numerical ray-tracer [37] has been developed for determining the optical appearance of distant objects in a given spacetime with an application to the Kerr-Newman spacetimes.

Another scheme for investigating physical situations with no known exact solutions is the post-Newtonian (PN) approximation [38,39], based on the assumptions of weak gravitational fields and slow motions. It provides a way to estimate nonlinear relativistic effects by using series expansion in inverse powers of the speed of light. This usually requires large algebraic computations where CAS can be used. Recently, Puetzfeld [40] developed a special package, PROCRUSTES, within Maple for PN calculations. The package supports the explicit determination of the geometric quantities, field equations, equations of motion, and conserved quantities in the PN approximation. A demo worksheet can reproduce many of the results of Chandrasekhar's classic paper [41].The package can be used to verify hand calculations or to generate the input for further numerical investigations.

Series expansion capability of CAS has also been used for the analysis of some problems with complicated closed form exact solutions. In the late eighties of the last century the present author used muMath for obtaining approximate expressions for the density, principal pressures and angular velociy of a fluid source for the Kerr metric [42]. Later, Mathematica was used to analyze the physical characteristics of composite fluid sources for the rotating Curzon metrics [43]. More recently, Mathematica has been used for obtaining a power series solution for the important problem of relativistic incompressible perfect fluid cylinders [44].

3. CAS in GR Teaching

In contrast to the early use of CAS in GR research, the use of CAS in GR teaching, or teaching in general, has not been possible until recently when less expensive and more powerful computing facilities became widely available. It first started at the post-graduate level and now it is a growing practice at the under-graduate level at many universities. Two experiences have been reported in the literature.

Ghergu and Vulcanov [45] reported that they have used CAS in teaching a GR course at the West University of Timisoara, Romania. They have used Reduce and Excalc [46] for algebraic computing, and Mathematica and Maple for graphic visualization. Using some simple Excalc procedures, they have been able to teach Riemannian geometry and how to obtain some exact solutions. They found that using a computer, students could quickly and comfortably learn the important notions of differential geometry, tensor calculus, and exterior calculus. Besides, they could easily obtain and analyze the Schwarzschild, the Reissner-Nordstrom, the de Sitter and the anti-de Sitter solutions.

Recently, Lake [47] developed GRTensorJ-Books [48], an interactive interface to GRTensorII for providing students of GR with an advanced calculator-style tool. All standard functions associated with a classical tensor approach are available as built-in functions. Metrics are referenced directly by equation numbers in ten widely used textbooks. When a student enters a metric equation number calculations are done in real time.

4. Conclusion

CAS have been quite useful in GR research. Many results would have not been obtained without using CAS. In the early years, authors used to refer to CAS almost every time they have used them. More recently, the use of CAS has become so common that such a reference is rarely made. CAS have been used in various GR areas, in particular:

- Finding, checking and analyzing exact solutions
- Classification of exact solutions
- Setting up databases of solutions
- Determination of hidden symmetries
- Solving matching problems
- Finding and analyzing approximate solutions
- Generating input formulas for numerical solutions and investigations.

The link between CAS and GR has been bi-directional, with CAS also benefiting from GR. The specific needs for GR have initiated the development of many systems and packages. It is noteworthy that many conferences and conference sessions have been organized to provide forum for discussing new computer algebra algorithms, techniques, software systems and applications in GR.

The use of CAS in GR teaching is growing. It is hoped that it will be a common practice in the near future.

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