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Math 410 (Prof. Bayly) EXAM 1: Monday 13 September 2004

There are 5 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

(1)(10 points) Let $A = \begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}$. Evaluate AA^T and A^TA, and show that

$$trace(A^TA) = trace(AA^T)$$

for any matrix of this form. Do you expect this to be true for matrices of other dimensions?

$$AA^T = \underset{2\times 3}{\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}} \underset{3\times 2}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} a^2+c^2+e^2 & ab+cd+ef \\ ba+dc+fe & b^2+d^2+f^2 \end{pmatrix}}$$

$$A^TA = \underset{3\times 2}{\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} a & c & e \\ b & d & f \end{pmatrix}} = \underset{3\times 3}{\begin{pmatrix} a^2+b^2 & ac+bd & ae+bf \\ ca+db & c^2+d^2 & ce+df \\ ea+fb & ec+fd & e^2+f^2 \end{pmatrix}}$$

$$\text{Trace } AA^T = a^2+c^2+e^2+b^2+d^2+f^2$$

$$\text{Trace } A^TA = a^2+b^2+c^2+d^2+e^2+f^2$$

SAME!

(e) YES! ~~Max value of~~ $\frac{\|A\vec{x}\|_1}{\|\vec{x}\|_1}$ is always ≤ 10, sometimes $= 10$ $\Rightarrow$ MAX VALUE $= 10$, (2)

(2)(20 points) The 1-norm of a vector $\vec{x} = (x_1, x_2, \cdots, x_n)^T$ is defined by $\|\vec{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

(a)(3 points) Find the 1-norms of the following vectors:

$$3 = \left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\| \quad , \quad \left\|\begin{pmatrix}0\\1\\0\end{pmatrix}\right\| = 1 \quad , \quad \left\|\begin{pmatrix}1\\-2\\-1\end{pmatrix}\right\| = 4$$

From here on we'll assume all vectors have nonnegative components, so we don't have to carry absolute value signs around.

(b)(5 points) Let

$$A = \begin{pmatrix}1 & 4 & 2\\3 & 5 & 1\\2 & 1 & 2\end{pmatrix} \quad , \quad \vec{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}.$$

Calculate the matrix-vector product $A\vec{x}$ and its 1-norm $\|A\vec{x}\|_1$ in terms of x, y, z.

(c)(5 points) Why is $\|A\vec{x}\|_1$ guaranteed less than or equal to $10\|\vec{x}\|_1$? (We're assuming x, y, z nonnegative, remember.)

(d)(5 points) Find a vector $\vec{x}$ for which $\|A\vec{x}\|_1 = 10\|\vec{x}\|_1$. (Hint: Most of its components can be chosen to be zero!)

(e)(2 points) Can you conclude that $\|A\|_1 = \max \frac{\|A\vec{x}\|_1}{\|\vec{x}\|_1} = 10$?

(b) $A\vec{x} = \begin{pmatrix}1 & 4 & 2\\3 & 5 & 1\\2 & 1 & 2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+4y+2z\\3x+5y+z\\2x+y+2z\end{pmatrix}$ All ≥ 0 since $x, y, z \geq 0$!

$\|A\vec{x}\|_1 = (x+4y+2z) + (3x+5y+z) + (2x+y+2z)$

$= 6x + 10y + 5z$

(c) If $x, y, z \geq 0$ then $10\underbrace{(x+y+z)}_{=\|\vec{x}\|_1} = 10x + 10y + 10z$ is $\geq 6x + 10y + 5z$

(d) If $x = 0$, $y =$ anything > 0, $z = 0$ then $6x + 10y + 5z = 10x + 10y + 10z$

so $\begin{pmatrix}0\\1\\0\end{pmatrix}$ works.

3

(3)(30 points) Consider the linear system

$$x + y + 2z = b_1 \quad , \quad 2x + y + 3z = b_2 \quad , \quad 4x + 3y + 7z = b_3 .$$

(a)(10 points) Write down the augmented matrix for this system, and use Gaussian elimination to reduce it to echelon form. Identify the lower and upper triangular factors L, U of the coefficient matrix, and check directly that $LU =$ *original matrix*.

(b)(15 points) Under what compatibility conditions on $\vec{b}$ does a solution exist? Find the general solution, identify the particular and complementary parts, and the kernel basis vector(s).

(c)(5 points) What are the dimensions of the range and kernel of the matrix in this problem?

(a) $$\left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 2 & 1 & 3 & b_2 \\ 4 & 3 & 7 & b_3 \end{array}\right) \xrightarrow[L_{31}=4]{L_{21}=2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & b_1 \\ 0 & -1 & -1 & b_2-2b_1 \\ 0 & -1 & -1 & b_3-4b_1 \end{array}\right)$$

$$\xrightarrow{L_{32}=1} \left(\begin{array}{ccc|c} \boxed{1}\text{ pivot} & 1 & 2 & b_1 \\ 0 & \boxed{-1}\text{ pivot} & -1 & b_2-2b_1 \\ 0 & 0 & 0 & b_3 - b_2 - 2b_1 \end{array}\right)$$

ECHELON FORM

RANK = 2 ✓

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad LU = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 4 & 3 & 7 \end{pmatrix}$$

(b) Since row 3 of U is all 0's $\Rightarrow$ solution exists only if $b_3 - b_2 - 2b_1 = 0$. If so, lets use Jordan elimination to put matrix in REDUCED echelon form:

3b cont

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & b_2-b_1 \\ 0 & 1 & 1 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad z \text{ free!}$$

$$\Rightarrow \quad y = 2b_1 - b_2 - z$$

$$x = b_2 - b_1 - z$$

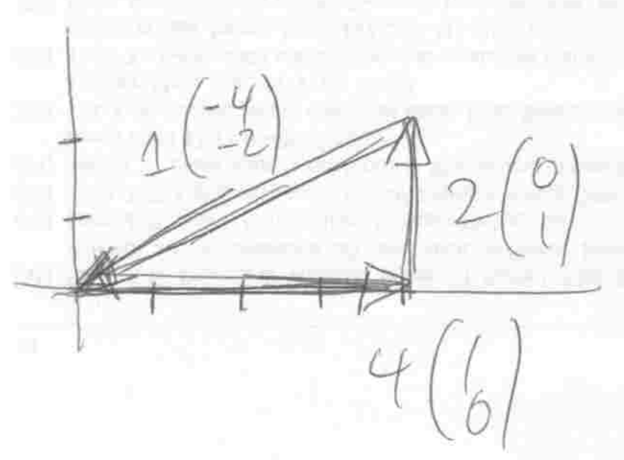
$$\vec{x} = \underbrace{\begin{pmatrix} 2b_1-b_2 \\ b_2-b_1 \\ 0 \end{pmatrix}}_{\text{particular}} + \underbrace{z\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}}_{\text{complementary}} \leftarrow \vec{k}$$

$\vec{k}$ = kernel basis vector

c) DIM RANGE = RANK = 2

Dim Kernel = # free variables = 1

5d) $\vec{k} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} \Rightarrow A\vec{k} = \vec{0} \quad 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -4 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$



5

(4)(20 points) Write the system of linear equations

$$x + ay = 1 \quad , \quad ax + 4y = 2$$

in matrix-vector form (a is assumed to be a given real number).

(a)(10 points) For what values of a are there no solutions, one unique solution, or many (i.e. an infinite number) solutions?

(b)(5 points) For the value of a giving no solutions, sketch the aiming vectors and the target vector, and indicate graphically why there is no solution.

(c)(5 points) For the value of a giving many solutions, sketch the aiming vectors and the target vector, and indicate graphically why there are many solutions.

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & a \\ a & 4 \end{pmatrix}$ $\quad$ ~~and~~ $A\vec{x} = \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Augmented matrix $\left(\begin{array}{cc|c} 1 & a & 1 \\ a & 4 & 2 \end{array}\right) \xrightarrow{l_{21} = a} \left(\begin{array}{cc|c} 1 & a & 1 \\ 0 & 4-a^2 & 2-a \end{array}\right)$

If $4 - a^2 \neq 0$ then ~~solve~~ $y = \dfrac{2-a}{4-a^2}$ is unique y,

and then $x = 1 - a\left(\dfrac{2-a}{4-a^2}\right)$ is unique x.

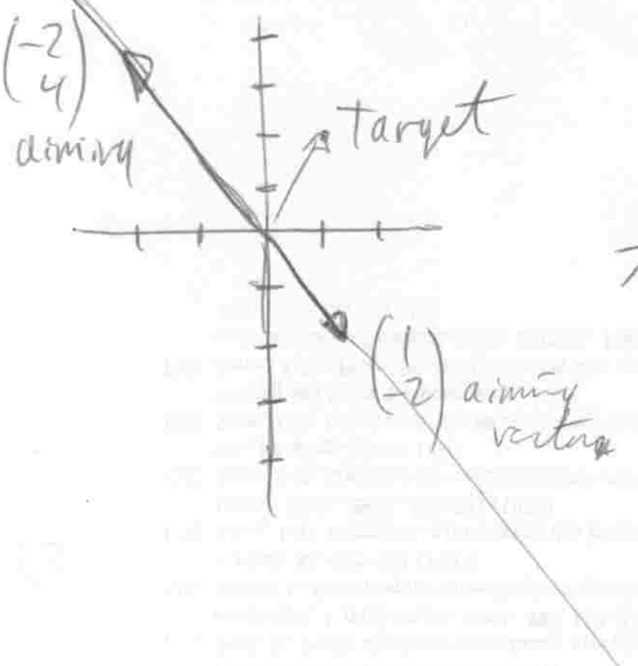
$\Rightarrow$ UNIQUE SOLUTION if $a^2 \neq 4$

If $a^2 = 4 \Rightarrow a = \pm 2$; need to consider separately

If $a = 2 \Rightarrow \left(\begin{array}{cc|c} 1 & \cancel{a}\,2 & 1 \\ 0 & 0 & 0 \end{array}\right) \leftarrow$ OK! $\quad y$ free, $x = 1 - \cancel{a}\,2y$ $\quad \infty$ solutions!

If $a = -2 \Rightarrow \left(\begin{array}{cc|c} 1 & \cancel{a}\,-2 & 1 \\ 0 & 0 & 4 \end{array}\right) \leftarrow$ NO SOLUTION!

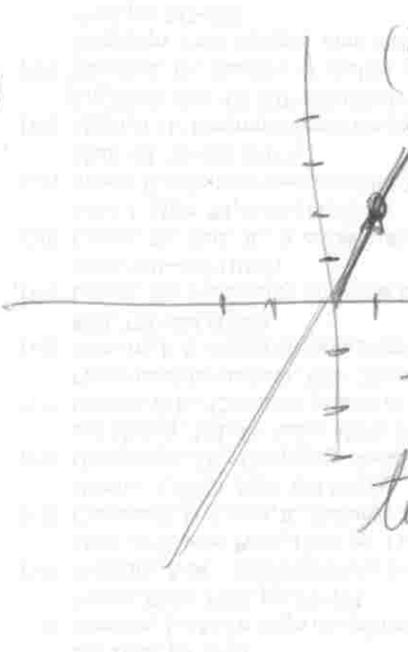
4b If $a = -2$ $\quad A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



Aiming vector can only hit target on line $y = -2x$

BUT $\vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is NOT on this line!

4c $\quad A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



aiming = target = $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

There are MANY ways to combine $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ & $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ to hit $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$: $1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 4 \end{pmatrix}$,

$3\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, etc.

(5)(20 points) Consider the system of linear equations $A\vec{x} = \vec{b}$, where

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \end{pmatrix} \quad , \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Note the matrix is already in reduced echelon form!

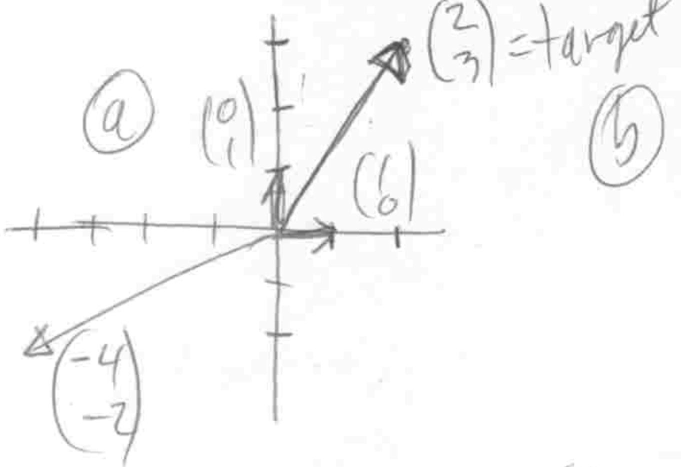
(a)(5 points) Sketch the aiming vectors (columns of A) and target vector $\vec{b}$.

(b)(5 points) Write down the general solution of this system, and identify the particular solution, complementary solution, free variable(s) and kernel basis vector(s).

(c)(5 points) Sketch the combination of aiming vectors that corresponds to the particular solution, and verify that it yields the target vector.

(d)(5 points) Sketch the combination of aiming vectors that corresponds to one of the kernel basis vector(s), and verify that it forms a closed cycle.

(a)



(b) z is free, $x - 4z = 2$

$\Rightarrow x = 2 + 4z$

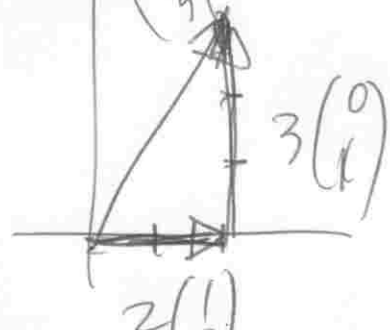
$y - 2z = 3 \Rightarrow$

$y = 3 + 2z$

$$\vec{x} = \begin{pmatrix} 2+4z \\ 3+2z \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}}_{\vec{x}^{(p)}} + \underbrace{z \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}}_{\vec{x}^{(c)}} \leftarrow \vec{k} = \text{kernel basis vector}$$

(c) $$A\vec{x}^{(p)} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} -4 \\ -2 \end{pmatrix}$$



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