Math 410 (Prof. Bayly) EXAM 3 : Friday 3 December 2004

There are 4 problems on this exam. They are not all the same length or difficulty, nor the same number of points. You should read through the entire exam before deciding which problems you will work on earlier or later. You are not expected to complete everything, but you should do as much as you can. It is *extremely* important to show your work!

No calculators are allowed on this exam. If your calculations become numerically awkward and time-consuming, you should describe the steps you would take if you had a calculator.

NOTE Problem 3 is rather lengthy; I have included a blank sheet between them to give you extra writing space.

(1)(25 points) Consider the matrices

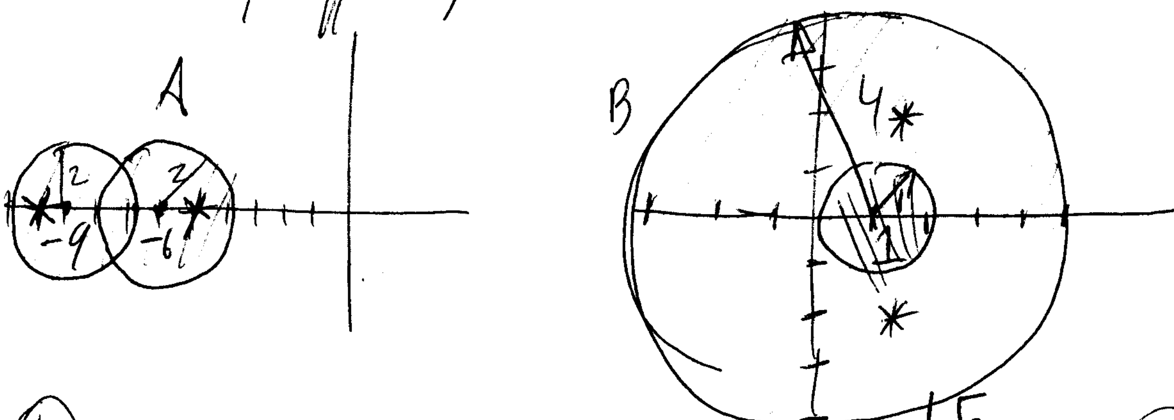
$$A = \begin{pmatrix} -6 & 2 \\ 2 & -9 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$$

(a)(5 points) BEFORE calculating the eigenvalues of A, identify and sketch a region of the complex plane guaranteed to contain the eigenvalues. Do the same for B.

(b)(10 points) Find the eigenvalues of A, and check that they are in the expected region.

(c)(10 points) Find the eigenvalues of B, and check that they are in the expected region.

(1) GERSHGORIN'S THEOREM : Eigenvalues lie in disks with centers = diagonal entries, radiuses = sum of absolute values of off diagonal entries in each column.



(b) Eigenvalues of A satisfy $\lambda^2 + \overset{15}{\cancel{54}}\lambda + \overbrace{(54-4)}^{50} = 0$

$(\lambda+5)(\lambda+10) = 0 \Rightarrow \boxed{\lambda = -5, -10}$ Yes! Inside Gershgorin disks!

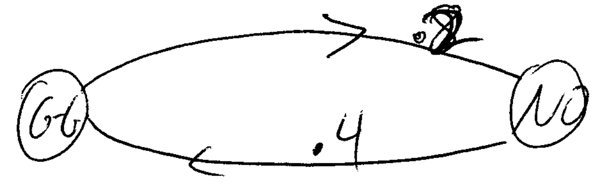
(c) For $B = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$ $\quad (-\lambda+1)^2 + 4 = 0$

$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{1}{2}\left[2 \pm \sqrt{4-20=-16}\right] = 1 \pm 2i$$

YES! These are inside the Gershgorin disks.

or



since we know that .8 & .6 will stay.

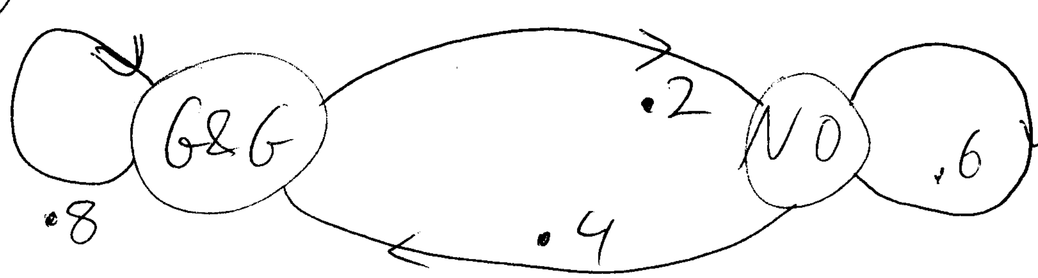
(2)(20 points) The *Goose and Gherkin* and *No Octopi* are neighboring restaurants that start the year with 75 customers each. The Goose regularly presents live music, with the result that 80 per cent of the patrons one night return on the next night, with the other 20 per cent going to No Octopi for some quiet pizza. Meanwhile 60 per cent of the customers at No Octopi return the next night, with 40 per cent going over to the Goose.

(a)(5 points) Draw a directed graph representing this situation (it can be a pretty rough sketch), and write down the Markov matrix of the system.

(b)(5 points) What are the expected numbers at the restaurants on the second day and the third day?

(c)(10 points) As weeks and weeks go by (i.e. as time goes to infinity), what are the expected numbers of customers at the two restaurants?

(a)



$M = \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix}$

(b) on day 1 population vector is $\begin{pmatrix} 75 \\ 75 \end{pmatrix} = \vec{p}(1)$

on day 2 p.v. $= \vec{p}(2) = M\vec{p}(1) = \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix}\begin{pmatrix} 75 \\ 75 \end{pmatrix} = \begin{pmatrix} 90 \\ 60 \end{pmatrix}$

on day 3 $\vec{p}(3) = M\vec{p}(2) = \begin{pmatrix} .8 & .4 \\ .2 & .6 \end{pmatrix}\begin{pmatrix} 90 \\ 60 \end{pmatrix} = \begin{pmatrix} 96 \\ 54 \end{pmatrix}$

(c) Find eigenvector belonging to $\lambda = 1$:

(over)

$(M - 1I)\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -.2 & .4 & 0 \\ \cancel{.2} & \cancel{-.4} & \cancel{0} \end{array}\right) \to 0's$ y free

$-.2x + .4y = 0 \Rightarrow x = 2y$

So $\vec{p}\,(\text{as } t \to \infty) = y \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We know total population $= 150$ always $\Rightarrow 2y + y = 150$

i.e. $3y = 150 \Rightarrow y = 50$, $x = 100$

So LONGTERM state is $\begin{pmatrix} 100 \\ 50 \end{pmatrix}$ $\leftarrow$ at G&G, $\leftarrow$ at N.O.

(3)(32 points) A model of guerrilla warfare assumes that the numbers $x(t), y(t)$ of combatants from sides X and Y satisfy

$$\frac{dx}{dt} = -y \quad , \quad \frac{dy}{dt} = -4x.$$

The difference in coefficients indicates that the X forces are 4 times as well-trained and equipped as the Y forces.

(a)(2 points) Express these equations as $\frac{d\vec{x}}{dt} = A\vec{x}$, where $\vec{x}(t) = (x(t), y(t))^T$ and A is the coefficient matrix that reproduces the above equations.

(b)(10 points) Find the eigenvalues and eigenvectors of A.

(c)(10 points) Find an invertible matrix S for which you expect (but don't have to check) $S^{-1}AS$ is diagonal. Then calculate the matrix exponential e^{tA}.

(c)(5 points) The general of X assumes that their factor of 4 advantage means that with a company of initial strength $x(0) = 100$ they can prevail over an adversary Y of initial strength $y(0) = 300$. Use e^{tA} to find $\vec{x}(t)$.

(d)(5 points) The battle ends when one of the variables becomes equal to zero. Which side will actually be the winner?

(a) $\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}$

(b) $\lambda^2 - 4 = 0 \Rightarrow \boxed{\lambda = \pm 2}$ eigenvalues

If $\lambda^{(1)} = 2$. then $(A - 2I)\vec{x} = \vec{0} \Rightarrow \left(\begin{array}{cc|c} -2 & -1 & 0 \\ \cancel{-4} & \cancel{-2} & \cancel{0} \end{array}\right) \to 0\text{'s}$

y free $\quad -2x - y = 0 \quad x = -\frac{1}{2}y$

$\vec{x} = y\begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$ choose $y = 2 \Rightarrow \vec{\xi}^{(1)} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

If $\lambda^{(2)} = -2$ then $(A + 2I)\vec{x} = \vec{0}$ $\left(\begin{array}{cc|c} 2 & -1 & 0 \\ \cancel{-4} & \cancel{2} & \cancel{0} \end{array}\right) \to 0\text{'s}$

(cont)

$2x - y = 0 \qquad x = \frac{1}{2}y \qquad \vec{x} = y\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

choose $y = 2 \Rightarrow \vec{\xi}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(c) $S = \begin{pmatrix} \vec{\xi}^{(1)} & \vec{\xi}^{(2)} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$

$$S^{-1} = \frac{1}{-2-2}\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}$$

<u>SO</u> $e^{tA} = S e^{t\Lambda} S^{-1}$

$$= \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\frac{1}{4}\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}$$

do this First

$$= \begin{pmatrix} -e^{2t} & e^{-2t} \\ 2e^{2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix}\frac{1}{4} = \cancel{\frac{1}{4}\Big(2e^{2t}}$$

$$= \frac{1}{4}\begin{pmatrix} 2e^{2t} + 2e^{-2t} & -e^{2t} + e^{-2t} \\ -4e^{2t} + 4e^{-2t} & 2e^{2t} + 2e^{-2t} \end{pmatrix} = e^{tA}$$

Extra workspace for problem 3!

second ⓒ $\vec{x}(t=0) = \begin{pmatrix} 100 \\ 300 \end{pmatrix}$ $\vec{x}(t) = e^{tA}\, \vec{x}(0)$

$$\Rightarrow \vec{x}(t) = \frac{1}{4}\begin{pmatrix} 2e^{2t}+2e^{-2t} & -e^{2t}+e^{-2t} \\ -4e^{2t}+4e^{-2t} & 2e^{2t}+2e^{-2t} \end{pmatrix}\begin{pmatrix} 100 \\ 300 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 200e^{2t}+200e^{-2t}-300e^{2t}+300e^{-2t} \\ -400e^{2t}+400e^{-2t}+600e^{2t}+600e^{-2t} \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} -100e^{2t}+500e^{-2t} \\ 200e^{2t}+1000e^{-2t} \end{pmatrix} = \begin{pmatrix} 125e^{-2t}-25e^{2t} \\ 50e^{2t}+250e^{-2t} \end{pmatrix}$$

ⓓ $y(t)$ is always > 0 so country Y is NOT loser

$x(t) = 0$ if $25e^{2t} = 125e^{-2t}$ $\Rightarrow X$ is the loser.

So Y wins despite lack of training & equipment, because they simply outnumber X initially.

(4)(25 points) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

(a)(10 points) Calculate AA^T and find its eigenvalues $\lambda^{(1)}, \lambda^{(2)}$. What are the singular values?

(b)(10 points) Find the corresponding eigenvectors $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}$ and check that they are orthogonal.

(c)(5 points) Find a matrix Q whose columns are unit vectors orthogonal to each other, and a diagonal matrix Λ, for which $AA^T = Q\Lambda Q^T$.

(a) $AA^T = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$

Char. poly $(5-\lambda)^2 - 1 = \lambda^2 - 10\lambda + 24 = (\lambda-4)(\lambda-6)$

$\Rightarrow \lambda = 4, 6$ For SVD choose $\lambda^{(1)} = 6$, $\lambda^{(2)} = 4$

SING VALUES $\sigma = \sqrt{\lambda} \Rightarrow \sigma^{(1)} = \sqrt{6}$, $\sigma^{(2)} = \sqrt{4} = 2$

(b) $\lambda^{(1)} = 6$ $(AA^T - 6I)\vec{x} = \vec{0}$ $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right)$ 0's y free

$-x + y = 0 \Rightarrow x = y$ $\vec{x} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ choose $y = 1$ for convenience

$\vec{\xi}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda^{(2)} = 4$ $(AA^T - 4I)\vec{x} = \vec{0}$ $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array}\right)$ 0's

(over)

y free again; $x+y=0 \Rightarrow x=-y$

$\vec{x} = y\begin{pmatrix}-1\\1\end{pmatrix}$ choose $y=1$ for convenience $\vec{\xi}^{(2)} = \begin{pmatrix}-1\\1\end{pmatrix}$

ORTHOGONAL? $\vec{\xi}^{(1)T}\vec{\xi}^{(2)} = \begin{pmatrix}1 & 1\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix} = -1+1=0$ ✓

YES!

(c) If we were using S matrix, $S=\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}$ would produce $AA^T = S\Lambda S^{-1}$. However if we make columns of S unit vectors, we obtain

$$\boxed{Q = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}}$$ where inverse is $Q^{-1}=Q^T$

~~and~~ with $\boxed{\Lambda = \begin{pmatrix}6 & 0\\0 & 4\end{pmatrix}}$

check (optional) $Q\Lambda Q^T = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\1 & 1\end{pmatrix}\begin{pmatrix}6 & 0\\0 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix}\frac{1}{\sqrt{2}}$

$= \frac{1}{2}\begin{pmatrix}6 & -4\\6 & 4\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}10 & 2\\2 & 10\end{pmatrix} = \begin{pmatrix}5 & 1\\1 & 5\end{pmatrix}$ ✓