Solutions — Chapter 10

ilin **10.1. Linear Iterative Systems.**

10.1.1. Suppose $u^{(0)} = 1$. Find $u^{(1)}, u^{(10)}$, and $u^{(20)}$ when (a) $u^{(k+1)} = 2\,u^{(k)}$, (b) $u^{(k+1)} = -.9\,u^{(k)}$, (c) $u^{(k+1)} = \mathrm{i}\,u^{(k)}$, (d) $u^{(k+1)} = (1 - 2\,\mathrm{i}\,)\,u^{(k)}$. Is the system (i) stable? (ii) asymptotically stable? (iii) unstable?

Solution:
(a) $u^{(1)} = 2, u^{(10)} = 1024$ and $u^{(20)} = 1048576$; unstable.
(b) $u^{(1)} = -.9, u^{(10)} = .348678$ and $u^{(20)} = .121577$; asymptotically stable.
(c) $u^{(1)} = \mathrm{i}\,, u^{(10)} = -1$ and $u^{(20)} = 1$; stable.
(d) $u^{(1)} = 1 - 2\,\mathrm{i}\,, u^{(10)} = 237 + 3116\,\mathrm{i}$ and $u^{(20)} = -9653287 + 1476984\,\mathrm{i}\,$; unstable.

10.1.2. A bank offers 3.25% interest compounded yearly. Suppose you deposit \$100. (a) Set up a linear iterative equation to represent your bank balance. (b) How much money do you have after 10 years? (c) What if the interest is compounded monthly?

Solution:
(a) $u^{(k+1)} = 1.0325\,u^{(k)}$, $u^{(0)} = 100$, where $u^{(k)}$ represents the balance after k years. $u^{(10)} = 1.0325^{10} \times 100 = 137.69$ dollars.
(b) $u^{(k+1)} = (1 + .0325/12)\,u^{(k)} = 1.002708\,u^{(k)}$, $\mathbf{u}^{(0)} = 100$, where $u^{(k)}$ represents the balance after k months. $u^{(120)} = (1 + .0325/12)^{120} \times 100 = 138.34$ dollars.

10.1.3. Show that the yearly balances of an account whose interest is compounded monthly satisfy a linear iterative system. How is the effective yearly interest rate determined from the original annual interest rate?

Solution: If r is the yearly interest rate, then $u^{(k+1)} = (1 + r/12)\,u^{(k)}$, where $u^{(k)}$ represents the balance after k months. Thus, the balance after m years is $v^{(m)} = u^{(12\,m)}$, and satisfies $v^{(m+1)} = (1 + r/12)^{12}\,v^{(m)}$. Then $(1 + r/12)^{12} = 1 + y$ where y is the effective annual interest rate.

10.1.4. Show that, as the time interval of compounding goes to zero, the bank balance after k years approaches an exponential function $e^{r\,k}\,a$, where r is the yearly interest rate and a the initial balance.

Solution: The balance coming from compounding n times per year is $\left(1 + \dfrac{r}{n}\right)^{n\,k} a \longrightarrow e^{r\,k}\,a$ by a standard calculus limit, [**3**].

10.1.5. Let $u(t)$ denote the solution to the linear ordinary differential equation $\dot{u} = \alpha\,u$, $u(0) = a$. Let $h > 0$ be fixed. Show that the sample values $u^{(k)} = u(k\,h)$ satisfy a linear iterative system. What is the coefficient λ? Compare the stability properties of the differential equation and the corresponding iterative system.

Solution: Since $u(t) = a\,e^{\alpha\,t}$ we have $u^{(k+1)} = u\big((k+1)\,h\big) = a\,e^{\alpha\,(k+1)\,h} = e^{\alpha\,h}\big(a\,e^{\alpha\,k\,h}\big) =$

$e^{\alpha h} u^{(k)}$, and so $\lambda = e^{\alpha h}$. The stability properties are the same: $|\,\alpha\,| < 1$ for asymptotic stability; $|\,\alpha\,| \leq 1$ for stability, $|\,\alpha\,| > 1$ for an unstable system.

10.1.6. For which values of λ does the scalar iterative system (10.2) have a periodic solution, meaning that $u^{(k+m)} = u^{(k)}$ for some m?

Solution: The solution $u^{(k)} = \lambda^k u^{(0)}$ is periodic of period m if and only if $\lambda^m = 1$, and hence λ is an m^{th} root of unity. Thus, $\lambda = e^{2\,\mathrm{i}\,k\,\pi/m}$ for $k = 0, 1, 2, \ldots m-1$. If k and m have a common factor, then the solution is of smaller period, and so the solutions of period exactly m are when k is relatively prime to m and λ is a primitive m^{th} root of unity, as defined in Exercise 5.7.7.

♠ 10.1.7. Investigate the solutions of the linear iterative equation when λ is a complex number with $|\,\lambda\,| = 1$, and look for patterns. You can take the initial condition $a = 1$ for simplicity and just plot the phase (argument) $\theta^{(k)}$ of the solution $u^{(k)} = e^{\mathrm{i}\,\theta^{(k)}}$.

Solution: If θ is a rational multiple of π, the solution is periodic, as in Exercise 10.1.6. When θ/π is irrational, the iterates eventually fill up (i.e., are dense in) the unit circle. ■ More?

10.1.8. Consider the iterative systems $u^{(k+1)} = \lambda\, u^{(k)}$ and $v^{(k+1)} = \mu\, v^{(k)}$, where $|\,\lambda\,| > |\,\mu\,|$. Prove that, for any nonzero initial data $u^{(0)} = a \neq 0$, $v^{(0)} = b \neq 0$, the solution to the first is eventually larger (in modulus) than that of the second: $|\,u^{(k)}\,| > |\,v^{(k)}\,|$, for $k \gg 0$.

Solution: $|\,u^{(k)}\,| = |\,\lambda\,|^k\,|\,a\,| > |\,v^{(k)}\,| = |\,\mu\,|^k\,|\,b\,|$ provided $k > \dfrac{\log|\,b\,| - \log|\,a\,|}{\log|\,\lambda\,| - \log|\,\mu\,|}$, where the inequality relies on the fact that $\log|\,\lambda\,| > \log|\,\mu\,|$.

10.1.9. Let λ, c be fixed. Solve the *affine* (or inhomogeneous linear) iterative equation

$$u^{(k+1)} = \lambda\, u^{(k)} + c, \qquad u^{(0)} = a. \tag{10.5}$$

Discuss the possible behaviors of the solutions. *Hint*: Write the solution in the form $u^{(k)} = u^\star + v^{(k)}$, where $u^\star$ is the equilibrium solution.

Solution: The equilibrium solution is $u^\star = c/(1-\lambda)$. Then $v^{(k)} = u^{(k)} - u^\star$ satisfies the homogeneous system $v^{(k+1)} = \lambda\, v^{(k)}$, and so $v^{(k)} = \lambda^k v^{(0)} = \lambda^k (a - u^\star)$. Thus, the solution to (10.5) is $u^{(k)} = \lambda^k (a - u^\star) + u^\star$. If $|\,\lambda\,| < 1$, then the equilibrium is asymptotically stable, with $u^{(k)} \to u^\star$ as $k \to \infty$; if $|\,\lambda\,| = 1$, it is stable, and solutions that start near $u^\star$ stay nearby; if $|\,\lambda\,| > 1$, it is unstable, and all non-equilibrium solutions become unbounded: $|\,u^{(k)}\,| \to \infty$.

10.1.10. A bank offers 5% interest compounded yearly. Suppose you deposit \$120 in the account each year. Set up an affine iterative equation (10.5) to represent your bank balance. How much money do you have after 10 years? After you retire in 50 years? After 200 years?

Solution: Let $u^{(k)}$ represent the balance after k years. Then $u^{(k+1)} = 1.05\, u^{(k)} + 120$, with $u^{(0)} = 0$. The equilibrium solution is $u^\star = -120/.05 = -2400$, and so after k years the balance is $u^{(k)} = (1.05^k - 1) \cdot 2400$. Then $u^{(10)} = \$1,509.35$, $u^{(50)} = \$25,121.76$, $u^{(200)} = \$4,149,979.40$.

10.1.11. Redo Exercise 10.1.10 in the case when the interest is compounded monthly and you deposit \$10 each month.

Solution: If $u^{(k)}$ represent the balance after k months, then $u^{(k+1)} = (1 + .05/12)\, u^{(k)} + 10$, $u^{(0)} = 0$. The balance after k months is $u^{(k)} = (1.0041667^k - 1) \cdot 2400$. Then $u^{(120)} = \$1,552.82$, $u^{(600)} = \$26,686.52$, $u^{(2400)} = \$5,177,417.44$.

♡ 10.1.12. Each spring the deer in Minnesota produce offspring at a rate of roughly 1.2 times the total population, while approximately 5% of the population dies as a result of predators and natural causes. In the fall hunters are allowed to shoot 3,600 deer. This winter the Department of Natural Resources (DNR) estimates that there are 20,000 deer. Set up an affine iterative equation (10.5) to represent the deer population each subsequent year. Solve the system and find the population in the next 5 years. How many deer in the long term will there be? Using this information, formulate a reasonable policy of how many deer hunting licenses the DNR should allow each fall, assuming one kill per license.

Solution: $u^{(k+1)} = 1.15\,u^{(k)} - 3600$, $u^{(0)} = 20000$. The equilibrium is $u^\star = 3600/.15 = 24{,}000$. Since $\lambda = 1.15 > 1$, the equilibrium is unstable; if the initial number of deer is less than the equilibrium, the population will decrease to zero, while if it is greater, then the population will increase without limit. Two possible options: ban hunting for 2 years until the deer reaches equilibrium of 24,000 and then permit at the current rate again. Or to keep the population at 20,000 allow hunting of only 3,000 deer per year. In both cases, the instability of the equilibrium makes it unlikely that the population will maintain a stable number, so constant monitoring of the deer population is required. (More realistic models incorporate nonlinear terms, and are less prone to such instabilities.)

10.1.13. Find the explicit formula for the solution to the following linear iterative systems:

(a) $u^{(k+1)} = u^{(k)} - 2\,v^{(k)},\ \ v^{(k+1)} = -2\,u^{(k)} + v^{(k)},\ \ u^{(0)} = 1,\ \ v^{(0)} = 0.$

(b) $u^{(k+1)} = u^{(k)} - \frac{2}{3}\,v^{(k)},\ \ v^{(k+1)} = \frac{1}{2}\,u^{(k)} - \frac{1}{6}\,v^{(k)},\ \ u^{(0)} = -2,\ \ v^{(0)} = 3.$

(c) $u^{(k+1)} = u^{(k)} - v^{(k)},\ \ v^{(k+1)} = -u^{(k)} + 5\,v^{(k)},\ \ u^{(0)} = 1,\ \ v^{(0)} = 0.$

(d) $u^{(k+1)} = \frac{1}{2}\,u^{(k)} + v^{(k)},\quad v^{(k+1)} = v^{(k)} - 2\,w^{(k)},\quad w^{(k+1)} = \frac{1}{3}\,w^{(k)},$
$u^{(0)} = 1,\quad v^{(0)} = -1,\quad w^{(0)} = 1.$

(e) $u^{(k+1)} = -u^{(k)} + 2\,v^{(k)} - w^{(k)},\ \ v^{(k+1)} = -6\,u^{(k)} + 7\,v^{(k)} - 4\,w^{(k)},$
$w^{(k+1)} = -6\,u^{(k)} + 6\,v^{(k)} - 4\,w^{(k)},\quad u^{(0)} = 0,\quad v^{(0)} = 1,\quad w^{(0)} = 3.$

Solution:

(a) $u^{(k)} = \dfrac{3^k + (-1)^k}{2},\ \ v^{(k)} = \dfrac{-3^k + (-1)^k}{2}.$

(b) $u^{(k)} = -\dfrac{20}{2^k} + \dfrac{18}{3^k},\ \ v^{(k)} = -\dfrac{15}{2^k} + \dfrac{18}{3^k}.$

(c) $u^{(k)} = \dfrac{(\sqrt5 + 2)(3 - \sqrt5)^k + (\sqrt5 - 2)(3 + \sqrt5)^k}{2\sqrt5},\quad v^{(k)} = \dfrac{(3 - \sqrt5)^k - (3 + \sqrt5)^k}{2\sqrt5}.$

(d) $u^{(k)} = -8 + \dfrac{27}{2^k} - \dfrac{18}{3^k},\ \ v^{(k)} = -4 + \dfrac{1}{3^{k-1}},\ \ w^{(k)} = \dfrac{1}{3^k}.$

(e) $u^{(k)} = 1 - 2^k,\ \ v^{(k)} = 1 + 2\,(-1)^k - 2^{k+1},\ \ w^{(k)} = 4\,(-1)^k - 2^k.$

10.1.14. Find the explicit formula for the general solution to the linear iterative systems with the following coefficient matrices:

(a) $\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$, (b) $\begin{pmatrix} -2 & 7 \\ -1 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} -3 & 2 & -2 \\ -6 & 4 & -3 \\ 12 & -6 & -5 \end{pmatrix}$, (d) $\begin{pmatrix} -\frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 1 & -1 & \frac{2}{3} \end{pmatrix}$.

Solution:

(a) $\mathbf{u}^{(k)} = c_1\,(-1 - \sqrt2)^k \begin{pmatrix} -\sqrt2 \\ 1 \end{pmatrix} + c_2\,(-1 + \sqrt2)^k \begin{pmatrix} \sqrt2 \\ 1 \end{pmatrix};$

$$(b)\quad \begin{aligned}\mathbf{u}^{(k)} &= c_1\left(\tfrac12+\tfrac{\sqrt3}{2}\,\mathrm{i}\right)^k\begin{pmatrix}\frac{5-\mathrm{i}\sqrt3}{2}\\ 1\end{pmatrix}+c_2\left(\tfrac12-\tfrac{\sqrt3}{2}\,\mathrm{i}\right)^k\begin{pmatrix}\frac{5+\mathrm{i}\sqrt3}{2}\\ 1\end{pmatrix}\\ &= a_1\begin{pmatrix}\frac52\cos\frac13 k\pi+\frac{\sqrt3}{2}\sin\frac13 k\pi\\ \cos\frac13 k\pi\end{pmatrix}+a_2\begin{pmatrix}\frac52\sin\frac13 k\pi-\frac{\sqrt3}{2}\cos\frac13 k\pi\\ \sin\frac13 k\pi\end{pmatrix};\end{aligned}$$

$$(c)\quad \mathbf{u}^{(k)} = c_1\begin{pmatrix}1\\2\\0\end{pmatrix}+c_2(-2)^k\begin{pmatrix}2\\3\\2\end{pmatrix}+c_3(-3)^k\begin{pmatrix}2\\3\\3\end{pmatrix};$$

$$(d)\quad \mathbf{u}^{(k)} = c_1\left(-\tfrac12\right)^k\begin{pmatrix}1\\1\\0\end{pmatrix}+c_2\left(-\tfrac13\right)^k\begin{pmatrix}1\\2\\1\end{pmatrix}+c_3\left(\tfrac16\right)^k\begin{pmatrix}0\\1\\2\end{pmatrix}.$$

10.1.15. The k^{th} *Lucas number* is defined as $L^{(k)}=\left(\dfrac{1+\sqrt5}{2}\right)^k+\left(\dfrac{1-\sqrt5}{2}\right)^k$.

(a) Explain why the Lucas numbers satisfy the Fibonacci iterative equation $L^{(k+2)}=L^{(k+1)}+L^{(k)}$. (b) Write down the first 7 Lucas numbers. (c) Prove that every Lucas number is a positive integer.

Solution: (a) It suffices to note that the Lucas numbers are the general Fibonacci numbers (10.16) when $a=L^{(0)}=2$, $b=L^{(1)}=1$. (b) $2,1,3,4,7,11,18$. (c) Because the first two are integers and so, by induction, $L^{(k+2)}=L^{(k+1)}+L^{(k)}$ is an integer whenever $L^{(k+1)},L^{(k)}$ are integers.

10.1.16. Prove that all the Fibonacci integers $u^{(k)}$, $k\geq 0$, can be found by just computing the first term in the Binet formula (10.17) and then rounding off to the nearest integer.

Solution:

The second summand satisfies $\left|\,-\dfrac{1}{\sqrt5}\left(\dfrac{1-\sqrt5}{2}\right)^k\,\right| < .448\times .62^k < .5$ for all $k\geq 0$. *Q.E.D.*

10.1.17. What happens to the Fibonacci integers $u^{(k)}$ if we go "backward in time", i.e., for $k<0$? How is $u^{(-k)}$ related to $u^{(k)}$?

Solution: $u^{(-k)}=(-1)^{k+1}u^{(k)}$. Indeed, since $\dfrac{1}{\frac{1+\sqrt5}{2}}=\dfrac{-1+\sqrt5}{2}$,

$$\begin{aligned}u^{(-k)} &= \frac{1}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^{-k}-\left(\frac{1-\sqrt5}{2}\right)^{-k}\right]=\frac{1}{\sqrt5}\left[\left(\frac{-1+\sqrt5}{2}\right)^{k}-\left(\frac{-1-\sqrt5}{2}\right)^{k}\right]\\ &= \frac{1}{\sqrt5}\left[(-1)^k\left(\frac{1-\sqrt5}{2}\right)^{-k}-(-1)^k\left(\frac{1+\sqrt5}{2}\right)^{k}\right]\\ &= \frac{(-1)^{k+1}}{\sqrt5}\left[\left(\frac{1+\sqrt5}{2}\right)^{k}-\left(\frac{1-\sqrt5}{2}\right)^{k}\right]=(-1)^{k+1}u^{(k)}.\end{aligned}$$

10.1.18. Use formula (10.20) to compute the k^{th} power of the following matrices:

(a) $\begin{pmatrix}5&2\\2&2\end{pmatrix}$, (b) $\begin{pmatrix}4&1\\-2&1\end{pmatrix}$, (c) $\begin{pmatrix}1&1\\-1&1\end{pmatrix}$, (d) $\begin{pmatrix}1&1&2\\1&2&1\\2&1&1\end{pmatrix}$, (e) $\begin{pmatrix}0&1&0\\0&0&1\\-1&0&2\end{pmatrix}$.

Solution:

$$(a)\ \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}^k = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 6^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix},$$

$$(b)\ \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}^k = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & 2^k \end{pmatrix} \begin{pmatrix} -2 & -1 \\ 1 & 1 \end{pmatrix},$$

$$(c)\ \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^k = \begin{pmatrix} -\,\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (1+\mathrm{i})^k & 0 \\ 0 & (1-\mathrm{i})^k \end{pmatrix} \begin{pmatrix} \frac{\mathrm{i}}{2} & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & \frac{1}{2} \end{pmatrix},$$

$$(d)\ \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^k \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

$$(e)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 2 \end{pmatrix}^k =$$

$$\begin{pmatrix} \frac{3-\sqrt5}{2} & \frac{3+\sqrt5}{2} & 1 \\ \frac{-1+\sqrt5}{2} & \frac{-1-\sqrt5}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt5}{2}\right)^k & 0 & 0 \\ 0 & \left(\frac{1+\sqrt5}{2}\right)^k & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{-5-3\sqrt5}{10} & \frac{-5-\sqrt5}{10} & \frac{5+2\sqrt5}{5} \\ \frac{-5+3\sqrt5}{10} & \frac{-5+\sqrt5}{10} & \frac{5-2\sqrt5}{5} \\ 1 & 1 & -1 \end{pmatrix}.$$

10.1.19. Use your answer from Exercise 10.1.18 to solve the following iterative systems:
(a) $u^{(k+1)} = 5\,u^{(k)} + 2\,v^{(k)},\ \ v^{(k+1)} = 2\,u^{(k)} + 2\,v^{(k)},\ \ u^{(0)} = -1,\ \ v^{(0)} = 0,$
(b) $u^{(k+1)} = 4\,u^{(k)} + v^{(k)},\ \ v^{(k+1)} = -\,2\,u^{(k)} + v^{(k)},\ \ u^{(0)} = 1,\ \ v^{(0)} = -3,$
(c) $u^{(k+1)} = u^{(k)} + v^{(k)},\ \ v^{(k+1)} = -\,u^{(k)} + v^{(k)},\ \ u^{(0)} = 0,\ \ v^{(0)} = 2,$
(d) $u^{(k+1)} = u^{(k)} + v^{(k)} + 2\,w^{(k)},\ \ v^{(k+1)} = u^{(k)} + 2\,v^{(k)} + w^{(k)},$
$w^{(k+1)} = 2\,u^{(k)} + v^{(k)} + w^{(k)},\ \ u^{(0)} = 1,\ \ v^{(0)} = 0,\ \ w^{(0)} = 1,$
(e) $u^{(k+1)} = v^{(k)},\ \ v^{(k+1)} = w^{(k)},\ \ w^{(k+1)} = -\,u^{(k)} + 2\,w^{(k)},\ \ u^{(0)} = 1,\ \ v^{(0)} = 0,\ \ w^{(0)} = 0.$

Solution: (a) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -\frac{2}{5}\,6^k \\ -\frac{1}{5} \end{pmatrix}$, (b) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3^k \\ -2^{k+1} \end{pmatrix}$,

(c) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \end{pmatrix} = \begin{pmatrix} -\,\mathrm{i} & \mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -\,\mathrm{i}\,(1+\mathrm{i})^k \\ (1-\mathrm{i})^k \end{pmatrix}$, (d) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{3}\,4^k \\ \frac{1}{3} \\ 0 \end{pmatrix}$,

(e) $\begin{pmatrix} u^{(k)} \\ v^{(k)} \\ w^{(k)} \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt5}{2} & \frac{3+\sqrt5}{2} & 1 \\ \frac{-1+\sqrt5}{2} & \frac{-1-\sqrt5}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{-5+3\sqrt5}{10}\left(\frac{1+\sqrt5}{2}\right)^k \\ \frac{-5+\sqrt5}{10}\left(\frac{1+\sqrt5}{2}\right)^k \\ -\,1 \end{pmatrix}$.

10.1.20. (a) Given initial data $\mathbf{u}^{(0)} = (\,1, 1, 1\,)^T$, explain why the resulting solution $\mathbf{u}^{(k)}$ to the system in Example 10.7 has all integer entries. (b) Find the coefficients c_1, c_2, c_3 in the explicit solution formula (10.18). (c) Check the first few iterates to convince yourself that the solution formula does, in spite of appearances, always give an integer value.

Solution: (a) Since the coefficient matrix T has all integer entries, its product $T\,\mathbf{u}$ with any vector with integer entries also has integer entries; (b) $c_1 = -2,\ \ c_2 = 3,\ \ c_2 = -3;$
(c) $\mathbf{u}^{(1)} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$, $\mathbf{u}^{(2)} = \begin{pmatrix} -26 \\ 10 \\ -2 \end{pmatrix}$, $\mathbf{u}^{(3)} = \begin{pmatrix} 76 \\ -32 \\ 16 \end{pmatrix}$, $\mathbf{u}^{(4)} = \begin{pmatrix} -164 \\ 76 \\ -44 \end{pmatrix}$, $\mathbf{u}^{(5)} = \begin{pmatrix} 304 \\ -152 \\ 88 \end{pmatrix}$.

10.1.21. (a) Show how to convert the higher order linear iterative equation
$$u^{(k+j)} = c_1 u^{(k+j-1)} + c_2 u^{(k+j-2)} + \cdots + c_j u^{(k)}$$
into a first order system $\mathbf{u}^{(k)} = T\mathbf{u}^{(k)}$. *Hint*: See Example 10.6.

(b) Write down initial conditions that guarantee a unique solution $u^{(k)}$ for all $k \geq 0$.

Solution: The vectors $\mathbf{u}^{(k)} = \left(u^{(k)}, u^{(k+1)}, \ldots, u^{(k+j-1)} \right)^T \in \mathbb{R}^j$ satisfy $\mathbf{u}^{(k+1)} = T\mathbf{u}^{(k)}$, where $T = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ c_j & c_{j-1} & c_{j-2} & c_{j-3} & \ldots & c_1 \end{pmatrix}$. The initial conditions are $\mathbf{u}^{(0)} = \mathbf{a} = \left(a_0, a_1, \ldots, a_{j-1} \right)^T$, and so $u^{(0)} = a_0, u^{(1)} = a_1, \ldots, u^{(j-1)} = a_{j-1}$.

10.1.22. Apply the method of Exercise 10.1.21 to solve the following iterative equations:

(a) $u^{(k+2)} = -u^{(k+1)} + 2u^{(k)}, \quad u^{(0)} = 1, \quad u^{(1)} = 2.$

(b) $12u^{(k+2)} = u^{(k+1)} + u^{(k)}, \quad u^{(0)} = -1, \quad u^{(1)} = 2.$

(c) $u^{(k+2)} = 4u^{(k+1)} + u^{(k)}, \quad u^{(0)} = 1, \quad u^{(1)} = -1.$

(d) $u^{(k+2)} = 2u^{(k+1)} - 2u^{(k)}, \quad u^{(0)} = 1, \quad u^{(1)} = 3.$

(e) $u^{(k+3)} = 2u^{(k+2)} + u^{(k+1)} - 2u^{(k)}, \quad u^{(0)} = 0, \quad u^{(1)} = 2, \quad u^{(2)} = 3.$

(f) $u^{(k+3)} = u^{(k+2)} + 2u^{(k+1)} - 2u^{(k)}, \quad u^{(0)} = 0, \quad u^{(1)} = 1, \quad u^{(2)} = 1.$

Solution: (a) $u^{(k)} = \frac{4}{3} - \frac{1}{3}(-2)^k$, (b) $u^{(k)} = \left(\frac{1}{3}\right)^{k-1} + \left(-\frac{1}{4}\right)^{k-1}$,

(c) $u^{(k)} = \dfrac{(5-3\sqrt{5})(2+\sqrt{5})^k + (5+3\sqrt{5})(2-\sqrt{5})^k}{10}$;

(d) $u^{(k)} = \left(\frac{1}{2} - \mathrm{i}\right)(1+\mathrm{i})^k + \left(\frac{1}{2} + \mathrm{i}\right)(1-\mathrm{i})^k = 2^{k/2}\left(\cos\frac{1}{4}k\pi + 2\sin\frac{1}{4}k\pi\right)$;

(e) $u^{(k)} = -\frac{1}{2} - \frac{1}{2}(-1)^k + 2^k$; (f) $u^{(k)} = -1 + \left(1 + (-1)^k\right) 2^{k/2-1}$.

♣ 10.1.23. Starting with $u^{(0)} = 0$, $u^{(1)} = 0$, $u^{(2)} = 1$, we define the sequence of *tri–Fibonacci integers* $u^{(k)}$ by adding the previous three to get the next one. For instance, $u^{(3)} = u^{(0)} + u^{(1)} + u^{(2)} = 1$. (a) Write out the next four tri–Fibonacci integers. (b) Find a third order iterative equation for the n^{th} tri–Fibonacci integer. (c) Find an explicit formula for the solution, using a computer to approximate the eigenvalues. (d) Do they grow faster than the usual Fibonacci numbers? What is their overall rate of growth?

Solution: (a) $u^{(k+3)} = u^{(k+2)} + u^{(k+1)} + u^{(k)}$; (b) $u^{(4)} = 2, u^{(5)} = 4, u^{(6)} = 7, u^{(7)} = 13$;

(c) $$\begin{aligned} u^{(k)} &\approx .183 \times 1.839^k + 2\,\mathrm{Re}\,(-.0914018 + .340547\,\mathrm{i})\,(-.419643 + .606291\,\mathrm{i})^k \\ &= .183 \times 1.839^k - .737353^k \left(.182804\cos 2.17623\,k + .681093\sin 2.17623\,k\right). \end{aligned}$$

♣ 10.1.24. Suppose that Fibonacci's rabbits only live for eight years, [**38**]. (a) Write out an iterative equation to describe the rabbit population. (b) Write down the first few terms. (c) Convert your equation into a first order iterative system using the method of Exercise 10.1.21. (d) At what rate does the rabbit population grow?

Solution:

(a) $u^{(k)} = u^{(k-1)} + u^{(k-2)} - u^{(k-8)}$.

(b) $0, 1, 1, 2, 3, 5, 8, 13, 21, 33, 53, 84, 134, 213, 339, 539, 857, 1363, 2167, \ldots$.

(c) $\mathbf{u}^{(k)} = \left(u^{(k)}, u^{(k+1)}, \ldots, u^{(k+7)} \right)^T$ satisfies $\mathbf{u}^{(k+1)} = A\mathbf{u}^{(k)}$ where the 8×8 coefficient matrix A has 1's on the superdiagonal, last row $(-1,0,0,0,0,0,1,1)$ and all other entries 0.

(*d*) The growth rate is given by largest eigenvalue in magnitude: $\lambda_1 = 1.59$, with $u^{(n)} \propto 1.59^n$. For more details, see [**38**].

10.1.25. Find the general solution to the iterative system $u_i^{(k+1)} = u_{i-1}^{(k)} + u_{i+1}^{(k)}$, $i = 1, \ldots, n$, where we set $u_0^{(k)} = u_{n+1}^{(k)} = 0$ for all k. *Hint*: Use Exercise 8.2.46.

Solution: $$\mathbf{u}_i^{(k)} = \sum_{j=1}^{n} c_j \left(2\cos\frac{j\pi}{n+1} \right)^k \sin\frac{i\,j\,\pi}{n+1}, \qquad i = 1, \ldots, n.$$

10.1.26. Prove that the curves $E_k = \{\, T^k \mathbf{x} \mid \|\mathbf{x}\| = 1 \,\}$, $k = 0,1,2,\ldots$, sketched in Figure 10.2 form a family of ellipses with the same principal axes. What are the semi-axes? *Hint*: Use Exercise 8.5.21.

Solution: The key observation is that coefficient matrix T is symmetric. Then, according to Exercise 8.5.21, the principal axes of the ellipse $E_1 = \{\, T\mathbf{x} \mid \|\mathbf{x}\| = 1 \,\}$ are the orthogonal eigenvectors of T. Moreover, T^k is also symmetric and has the same eigenvectors. Hence, all the ellipses E_k have the same principal axes. The semi-axes are the absolute values of the eigenvalues, and hence E_k has semi-axes $(.8)^k$ and $(.4)^k$.

♠ 10.1.27. Plot the ellipses $E_k = \{\, T^k \mathbf{x} \mid \|\mathbf{x}\| = 1 \,\}$ for $k = 1,2,3,4$ for the following matrices T. Then determine their principal axes, semi-axes, and areas. *Hint*: Use Exercise 8.5.21.

$$(a)\ \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}, \qquad (b)\ \begin{pmatrix} 0 & -1.2 \\ .4 & 0 \end{pmatrix}, \qquad (c)\ \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix}.$$

Solution:

(*a*) ■itz1■

E_1: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, semi-axes: $1, \frac{1}{3}$, area: $\frac{1}{3}\pi$.

E_2: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, semi-axes: $1, \frac{1}{9}$, area: $\frac{1}{9}\pi$.

E_3: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, semi-axes: $1, \frac{1}{27}$, area: $\frac{1}{27}\pi$.

E_4: principal axes: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, semi-axes: $1, \frac{1}{81}$, area: $\frac{1}{81}\pi$.

(*b*) ■itz2■

E_1: principal axes: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, semi-axes: $1.2, .4$, area: $.48\pi = 1.5080$.

E_2: circle of radius $.48$, area: $.2304\pi = .7238$.

E_3: principal axes: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, semi-axes: $.576, .192$, area: $.1106\pi = .3474$.

E_4: circle of radius $.2304$, area: $.0531\pi = .1168$.

(*c*) ■itz3■

E_1: principal axes: $\begin{pmatrix} .6407 \\ .7678 \end{pmatrix}, \begin{pmatrix} -.7678 \\ .6407 \end{pmatrix}$, semi-axes: $1.0233, .3909$, area: $.4\pi = 1.2566$.

E_2: principal axes: $\begin{pmatrix} .6765 \\ .7365 \end{pmatrix}, \begin{pmatrix} -.7365 \\ .6765 \end{pmatrix}$, semi-axes: $1.0394, .1539$, area: $.16\pi = .5027$.

E_3: principal axes: $\begin{pmatrix} .6941 \\ .7199 \end{pmatrix}, \begin{pmatrix} -.7199 \\ .6941 \end{pmatrix}$, semi-axes: $1.0477, .0611$, area: $.064\pi = .2011$.

E_4: principal axes: $\begin{pmatrix} .7018 \\ .7124 \end{pmatrix}, \begin{pmatrix} -.7124 \\ .7018 \end{pmatrix}$, semi-axes: $1.0515, .0243$, area: $.0256\,\pi = .0804$.

10.1.28. Let T be a positive definite 2×2 matrix. Let $E_n = \{\, T^n \mathbf{x} \mid \| \mathbf{x} \| = 1 \,\}$, $n = 0, 1, 2, \ldots$, be the image of the unit circle under the n^{th} power of T. (a) Prove that E_n is an ellipse. *True or false*: (b) The ellipses E_n all have the same principal axes. (c) The semi-axes are given by $r_n = r_1^n, s_n = s_1^n$. (d) The areas are given by $A_n = \pi \alpha^n$ where $\alpha = A_1/\pi$.

Solution: (a) See Exercise 8.5.21. (b) True — they are the eigenvectors of T. (c) True — r_1, s_1 are the eigenvalues of T. (d) True, since the area is π times the product of the semi-axes, so $A_1 = \pi r_1 s_1$, so $\alpha = r_1 s_1 = |\det T\,|$. Then $A_n = \pi r_n s_n = \pi r_1^n s_1^n = \pi |\det T\,|^n = \pi\alpha^n$.

10.1.29. Answer Exercise 10.1.28 when T an arbitrary nonsingular 2×2 matrix. *Hint*: Use Exercise 8.5.21.

Solution: (a) Follows from Exercise 8.5.21. (b) False; see Exercise 10.1.27(c) for a counterexample. (c) False — the singular values of T^n are not, in general, the n^{th} powers of the singular values of T. (d) True, since the product of the singular values is the absolute value of the determinant, $A_n = \pi |\det T\,|^n$.

10.1.30. Given the general solution (10.9) of the iterative system $\mathbf{u}^{(k+1)} = T\,\mathbf{u}^{(k)}$, write down the solution to $\mathbf{v}^{(k+1)} = \alpha\, T\,\mathbf{v}^{(k)} + \beta\,\mathbf{v}^{(k)}$, where α, β are fixed scalars.

Solution: $\mathbf{v}^{(k)} = c_1 (\alpha\lambda_1 + \beta)^k\, \mathbf{v}_1 + \;\cdots\; + c_n (\alpha\lambda_n + \beta)^k\, \mathbf{v}_n$.

♢ 10.1.31. Prove directly that if the coefficient matrix of a linear iterative system is real, both the real and imaginary parts of a complex solution are real solutions.

Solution: If $\mathbf{u}^{(k)} = \mathbf{x}^{(k)} + \mathrm{i}\,\mathbf{y}^{(k)}$ is a complex solution, then the iterative equation becomes $\mathbf{x}^{(k+1)} + \mathrm{i}\,\mathbf{y}^{(k+1)} = T\,\mathbf{x}^{(k)} + \mathrm{i}\,T\,\mathbf{y}^{(k)}$. Separating the real and imaginary parts of this complex vector equation and using the fact that T is real, we deduce $\mathbf{x}^{(k+1)} = T\,\mathbf{x}^{(k)}$, $\mathbf{y}^{(k+1)} = T\,\mathbf{y}^{(k)}$. Therefore, $\mathbf{x}^{(k)}, \mathbf{y}^{(k)}$ are real solutions to the iterative system. *Q.E.D.*

♢ 10.1.32. Explain why the solution $\mathbf{u}^{(k)}$, $k \geq 0$, to the initial value problem (10.6) exists and is uniquely defined. Does this hold if we allow negative $k < 0$?

Solution: The formula uniquely specifies $\mathbf{u}^{(k+1)}$ once $\mathbf{u}^{(k)}$ is known. Thus, by induction, once the initial value $\mathbf{u}^{(0)}$ is fixed, there is only one possible solution for $k = 0, 1, 2, \ldots$. Existence and uniqueness also hold for $k < 0$ when T is nonsingular, since $\mathbf{u}^{(-k-1)} = T^{-1}\mathbf{u}^{(-k)}$. If T is singular, the solution will not exist for $k < 0$ if any $\mathbf{u}^{(-k)} \notin \operatorname{rng} T$, or, if it exists, is not unique since we can add any element of $\ker T$ to $\mathbf{u}^{(-k)}$ without affecting $\mathbf{u}^{(-k+1)}, \mathbf{u}^{(-k+2)}, \ldots$.

10.1.33. Prove that if T is a symmetric matrix, then the coefficients in (10.9) are given by the formula $c_j = \mathbf{a}^T \mathbf{v}_j \,/\, \mathbf{v}_j^T \mathbf{v}_j$.

Solution: According to Theorem 8.20, the eigenvectors of T are real and form an orthogonal basis of $\mathbb{R}^n$ with respect to the Euclidean norm. The formula for the coefficients c_j thus follows directly from (5.8).

10.1.34. Explain why the j^{th} column $\mathbf{c}_j^{(k)}$ of the matrix power T^k satisfies the linear iterative system $\mathbf{c}_j^{(k+1)} = T\,\mathbf{c}_j^{(k)}$ with initial data $\mathbf{c}_j^{(0)} = \mathbf{e}_j$, the j^{th} standard basis vector.

Solution: Since matrix multiplication acts column-wise, cf. (1.11), the j^{th} column of the matrix equation $T^{k+1} = T\,T^k$ is $\mathbf{c}_j^{(k+1)} = T\,\mathbf{c}_j^{(k)}$. Moreover, $T^0 = \mathrm{I}$ has j^{th} column $\mathbf{c}_j^{(0)} = \mathbf{e}_j$. *Q.E.D.*

10.1.35. Let $z^{(k+1)} = \lambda\, z^{(k)}$ be a complex scalar iterative equation with $\lambda = \mu + \mathrm{i}\,\nu$. Show that its real and imaginary parts $x^{(k)} = \operatorname{Re} z^{(k)}$, $y^{(k)} = \operatorname{Im} z^{(k)}$, satisfy a two-dimensional real linear iterative system. Use the eigenvalue method to solve the real 2×2 system, and verify that your solution coincides with the solution to the original complex equation.

Solution: $\begin{pmatrix} x^{(k+1)} \\ y^{(k+1)} \end{pmatrix} = \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix}$. The eigenvalues of the coefficient matrix are $\mu \pm \mathrm{i}\,\nu$, with eigenvectors $\begin{pmatrix} 1 \\ \mp\,\mathrm{i} \end{pmatrix}$ and so the solution is

$$\begin{pmatrix} x^{(k)} \\ y^{(k)} \end{pmatrix} = \frac{x^{(0)} + \mathrm{i}\,y^{(0)}}{2}\,(\mu + \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ -\,\mathrm{i} \end{pmatrix} + \frac{x^{(0)} - \mathrm{i}\,y^{(0)}}{2}\,(\mu - \mathrm{i}\,\nu)^k \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}.$$

Therefore $z^{(k)} = x^{(k)} + \mathrm{i}\,y^{(k)} = (x^{(0)} + \mathrm{i}\,y^{(0)})\,(\mu + \mathrm{i}\,\nu)^k = \lambda^k z^{(0)}$. *Q.E.D.*

♢ 10.1.36. Let T be an incomplete matrix, and suppose $\mathbf{w}_1, \ldots, \mathbf{w}_j$ is a Jordan chain associated with an incomplete eigenvalue λ. (*a*) Prove that, for any $i = 1, \ldots, j$,

$$T^k\,\mathbf{w}_i = \lambda^k\,\mathbf{w}_i + k\,\lambda^{k-1}\,\mathbf{w}_{i-1} + \binom{k}{2}\lambda^{k-2}\,\mathbf{w}_{i-2} + \;\cdots\;. \tag{10.23}$$

(*b*) Explain how to use a Jordan basis of T to construct the general solution to the linear iterative system $\mathbf{u}^{(k+1)} = T\,\mathbf{u}^{(k)}$.

Solution:

(*a*) Proof by induction:

$$\begin{aligned} T^{k+1}\,\mathbf{w}_i &= T\left(\lambda^k\,\mathbf{w}_i + k\,\lambda^{k-1}\,\mathbf{w}_{i-1} + \binom{k}{2}\lambda^{k-2}\,\mathbf{w}_{i-2} + \;\cdots\;\right) \\ &= \lambda^k\,T\,\mathbf{w}_i + k\,\lambda^{k-1}\,T\,\mathbf{w}_{i-1} + \binom{k}{2}\lambda^{k-2}\,T\,\mathbf{w}_{i-2} + \;\cdots \\ &= \lambda^k\,(\lambda\,\mathbf{w}_i + w_{i-1}) + k\,\lambda^{k-1}\,(\lambda\,\mathbf{w}_{i-1} + w_{i-2}) + \binom{k}{2}\lambda^{k-2}\,(\lambda\,\mathbf{w}_{i-2} + w_{i-3}) + \;\cdots \\ &= \lambda^{k+1}\,\mathbf{w}_i + (k+1)\,\lambda^k\,\mathbf{w}_{i-1} + \binom{k+1}{2}\lambda^{k-1}\,\mathbf{w}_{i-2} + \;\cdots\;. \end{aligned}$$

(*b*) Each Jordan chain of length j is used to construct j linearly independent solutions by formula (10.23). Thus, for an n-dimensional system, the Jordan basis produces the required number of linearly independent (complex) solutions, and the general solution is obtained by taking linear combinations. Real solutions of a real iterative system are obtained by using the real and imaginary parts of the Jordan chain solutions corresponding to the complex conjugate pairs of eigenvalues.

10.1.37. Use the method Exercise 10.1.36 to find the general real solution to the following iterative systems:

(*a*) $u^{(k+1)} = 2\,u^{(k)} + 3\,v^{(k)}, \;\; v^{(k+1)} = 2\,v^{(k)}$,

(*b*) $u^{(k+1)} = u^{(k)} + v^{(k)}, \;\; v^{(k+1)} = -\,4\,u^{(k)} + 5\,v^{(k)}$,

(*c*) $u^{(k+1)} = -\,u^{(k)} + v^{(k)} + w^{(k)}, \;\; v^{(k+1)} = -\,v^{(k)} + w^{(k)}, \;\; w^{(k+1)} = -\,w^{(k)}$,

(*d*) $u^{(k+1)} = 3\,u^{(k)} - v^{(k)}, \;\; v^{(k+1)} = -\,u^{(k)} + 3\,v^{(k)} + w^{(k)}, \;\; w^{(k+1)} = -\,v^{(k)} + 3\,w^{(k)}$,

(*e*) $u^{(k+1)} = u^{(k)} - v^{(k)} - w^{(k)}, \;\; v^{(k+1)} = 2\,u^{(k)} + 2\,v^{(k)} + 2\,w^{(k)}, \;\; w^{(k+1)} = -\,u^{(k)} + v^{(k)} + w^{(k)}$,

(*f*) $u^{(k+1)} = v^{(k)} + z^{(k)}, \;\; v^{(k+1)} = -\,u^{(k)} + w^{(k)}, \;\; w^{(k+1)} = z^{(k)}, \;\; z^{(k+1)} = -\,w^{(k)}$.

Solution:

(*a*) $u^{(k)} = 2^k\left(c_1 + \frac{1}{2}\,k\,c_2\right), \;\; v^{(k)} = \frac{1}{3}\,2^k\,c_2$;

(b) $u^{(k)} = 3^k\left[\,c_1 + \left(\frac{1}{3}\,k - \frac{1}{2}\right)c_2\,\right],\ \ v^{(k)} = 3^k\left[\,2\,c_1 + \frac{2}{3}\,k\,c_2\,\right];$

(c) $u^{(k)} = (-1)^k\left[\,c_1 - k\,c_2 + \frac{1}{2}\,k(k-1)\,c_3\,\right],\ \ v^{(k)} = (-1)^k\left[\,c_2 - (k+1)\,c_3\,\right],\ \ w^{(k)} = (-1)^k\,c_3;$

(d) $u^{(k)} = 3^k\left[\,c_1 + \frac{1}{3}\,k\,c_2 + \left(\frac{1}{18}\,k\,(k-1)+1\right)c_3\,\right],\ \ v^{(k)} = -3^k\left[\,c_2 + \frac{1}{3}\,k\,c_3\,\right],$
$w^{(k)} = 3^k\left[\,c_1 + \frac{1}{3}\,k\,c_2 + \frac{1}{18}\,k\,(k-1)\,c_3\,\right];$

(e) $u^{(0)} = -\,c_2,\ \ v^{(0)} = -\,c_1 + c_3,\ \ w^{(k)} = c_1 + c_2$, while, for $k > 0$,
$u^{(k)} = -\,2^k\left(\,c_2 + \frac{1}{2}\,k\,c_3\,\right),\ \ v^{(k)} = 2^k\,c_3,\ \ w^{(k)} = 2^k\left(\,c_2 + \frac{1}{2}\,k\,c_3\,\right);$

(f) $u^{(k)} = -\,\mathrm{i}^{k+1}\,c_1 - k\,\mathrm{i}^{k}\,c_2 - (-\,\mathrm{i}\,)^{k+1}c_3 - k(-\,\mathrm{i}\,)^k c_4, \qquad w^{(k)} = -\,\mathrm{i}^{k+1}\,c_2 - (-\,\mathrm{i}\,)^{k+1}c_4,$
$v^{(k)} = \mathrm{i}^{k}c_1 + k\,\mathrm{i}^{k-1}\,c_2 + (-\,\mathrm{i}\,)^k c_3 + k(-\,\mathrm{i}\,)^{k-1}c_4, \qquad z^{(k)} = \mathrm{i}^{k}\,c_2 + (-\,\mathrm{i}\,)^k c_4.$

10.1.38. Find a formula for the k^{th} power of a Jordan block matrix. *Hint*: Use Exercise 10.1.36.

Solution: $$J_{\lambda,n}^k = \begin{pmatrix} \lambda^k & k\,\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \binom{k}{3}\lambda^{k-3} & \ldots & \binom{k}{n-1}\lambda^{k-n+1} \\ 0 & \lambda^k & k\,\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \ldots & \binom{k}{n-2}\lambda^{k-n+2} \\ 0 & 0 & \lambda^k & k\,\lambda^{k-1} & \ldots & \binom{k}{n-3}\lambda^{k-n+3} \\ 0 & 0 & 0 & \lambda^k & \ldots & \binom{k}{n-4}\lambda^{k-n+4} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda^n \end{pmatrix}.$$

♡ 10.1.39. An *affine iterative system* has the form $\mathbf{u}^{(k+1)} = T\,\mathbf{u}^{(k)} + \mathbf{b},\ \ \mathbf{u}^{(0)} = \mathbf{c}$. ($a$) Under what conditions does the system have an equilibrium solution $\mathbf{u}^{(k)} \equiv \mathbf{u}^\star$? ($b$) In such cases, find a formula for the general solution. *Hint*: Look at $\mathbf{v}^{(k)} = \mathbf{u}^{(k)} - \mathbf{u}^\star$. ($c$) Solve the following affine iterative systems:

(i) $\mathbf{u}^{(k+1)} = \begin{pmatrix} 6 & 3 \\ -3 & -4 \end{pmatrix}\mathbf{u}^{(k)} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}^{(0)} = \begin{pmatrix} 4 \\ -3 \end{pmatrix},$

(ii) $\mathbf{u}^{(k+1)} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}\mathbf{u}^{(k)} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$

(iii) $\mathbf{u}^{(k+1)} = \begin{pmatrix} -3 & 2 & -2 \\ -6 & 4 & -3 \\ 12 & -6 & -5 \end{pmatrix}\mathbf{u}^{(k)} + \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \quad \mathbf{u}^{(0)} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$

(iv) $\mathbf{u}^{(k+1)} = \begin{pmatrix} -\frac{5}{6} & \frac{1}{3} & -\frac{1}{6} \\ 0 & -\frac{1}{2} & \frac{1}{3} \\ 1 & -1 & \frac{2}{3} \end{pmatrix}\mathbf{u}^{(k)} + \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{2} \end{pmatrix}, \quad \mathbf{u}^{(0)} = \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$ (d) Discuss what happens in cases when there is no fixed point, assuming that T is complete.

Solution:

(a) The system has an equilibrium solution if and only if $(T - \mathrm{I}\,)\mathbf{u}^\star = \mathbf{b}$. In particular, if 1 is not an eigenvalue of T, every $\mathbf{b}$ leads to an equilibrium solution.

(b) Since $\mathbf{v}^{(k+1)} = T\,\mathbf{v}^{(k)}$, the general solution is
$$\mathbf{u}^{(k)} = \mathbf{u}^\star + c_1\,\lambda_1^k\,\mathbf{v}_1 + c_2\,\lambda_2^k\,\mathbf{v}_2 + \ \cdots\ + c_n\,\lambda_n^k\,\mathbf{v}_n,$$
where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues of T.

(c) (i) $\mathbf{u}^{(k)} = \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix} - 5^k\begin{pmatrix} -3 \\ 1 \end{pmatrix} - (-3)^k\begin{pmatrix} -\frac{1}{3} \\ 1 \end{pmatrix};$

(ii) $\mathbf{u}^{(k)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \dfrac{(-1-\sqrt{2})^k}{2\sqrt{2}} \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} - \dfrac{(-1+\sqrt{2})^k}{2\sqrt{2}} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix};$

(iii) $\mathbf{u}^{(k)} = \begin{pmatrix} -1 \\ -\frac{3}{2} \\ -1 \end{pmatrix} - 3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{15}{2}\,(-2)^k \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - 5\,(-3)^k \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix};$

(iv) $\mathbf{u}^{(k)} = \begin{pmatrix} \frac{1}{6} \\ \frac{5}{3} \\ \frac{3}{2} \end{pmatrix} + \frac{7}{2}\left(-\frac{1}{2}\right)^k \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \frac{7}{2}\left(-\frac{1}{3}\right)^k \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \frac{7}{3}\left(\frac{1}{6}\right)^k \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}.$

(d) In general, using induction, the solution is

$$\mathbf{u}^{(k)} = T^k \mathbf{c} + (\,\mathrm{I} + T + T^2 + \cdots + T^{k-1})\mathbf{b}.$$

If we write $\mathbf{b} = b_1\,\mathbf{v}_1 + \cdots + b_n\,\mathbf{v}_n$, $\mathbf{c} = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n$, in terms of the eigenvectors, then

$$\mathbf{u}^{(k)} = \sum_{j=1}^{n} \left[\,\lambda_j^k c_j + (1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1}) b_j\,\right] \mathbf{v}_j.$$

If $\lambda_j \neq 1$, one can use the geometric sum formula $1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1} = \dfrac{1-\lambda_j^k}{1-\lambda_j}$, while if $\lambda_j = 1$, then $1 + \lambda_j + \lambda_j^2 + \cdots + \lambda_j^{k-1} = k$. Incidentally, the equilibrium solution is

$$\mathbf{u}^\star = \sum_{\lambda_j \neq 1} \frac{b_j}{1-\lambda_j}\,\mathbf{v}_j.$$

♣ 10.1.40. A well-known method of generating a sequence of "pseudo-random" integers $x_0, x_1, \ldots$ in the interval from 0 to n is based on the Fibonacci equation $u^{(k+2)} = u^{(k+1)} + u^{(k)} \bmod n$, with the initial values $u^{(0)}, u^{(1)}$ chosen from the integers $0, 1, 2, \ldots, n-1$. (a) Generate the sequence of pseudo-random numbers that result from the choices $n = 10$, $u^{(0)} = 3, u^{(1)} = 7$. Keep iterating until the sequence starts repeating. (b) Experiment with other sequences of pseudo-random numbers generated by the method.

Solution: ■

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listability

10.2. Stability.

10.2.1. Determine the spectral radius of the following matrices:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} \frac{1}{3} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix}$, (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix}$, (d) $\begin{pmatrix} -1 & 5 & -9 \\ 4 & 0 & -1 \\ 4 & -4 & 3 \end{pmatrix}$.

Solution:

(a) Eigenvalues: $\frac{5+\sqrt{33}}{2} \approx 5.3723$, $\frac{5-\sqrt{33}}{2} \approx -.3723$; spectral radius: $\frac{5+\sqrt{33}}{2} \approx 5.3723$.

(b) Eigenvalues: $\pm \frac{\mathrm{i}}{6\sqrt{2}} \approx \pm .11785\,\mathrm{i}$; spectral radius: $\frac{1}{6\sqrt{2}} \approx .11785$.

(c) Eigenvalues: $2, 1, -1$; spectral radius: 2.

(d) Eigenvalues: $4, -1 \pm 4\,\mathrm{i}$; spectral radius: $\sqrt{17} \approx 4.1231$.

10.2.2. Determine whether or not the following matrices are convergent:

$$(a)\ \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}, \quad (b)\ \begin{pmatrix} .6 & .3 \\ .3 & .7 \end{pmatrix}, \quad (c)\ \frac{1}{5}\begin{pmatrix} 5 & -3 & -2 \\ 1 & -2 & 1 \\ 1 & -5 & 4 \end{pmatrix}, \quad (d)\ \begin{pmatrix} .8 & .3 & .2 \\ .1 & .2 & .6 \\ .1 & .5 & .2 \end{pmatrix}.$$

Solution:

(a) Eigenvalues: $2 \pm 3\,\mathrm{i}$; spectral radius: $\sqrt{13} \approx 3.6056$; not convergent.
(b) Eigenvalues: $.95414, .34586$; spectral radius: $.95414$; convergent.
(c) Eigenvalues: $\frac{4}{5}, \frac{3}{5}, 0$; spectral radius: $\frac{4}{5}$; convergent.
(d) Eigenvalues: $1., .547214, -.347214$; spectral radius: 1; not convergent.

10.2.3. Which of the listed coefficient matrices defines a linear iterative system with asymptotically stable zero solution?

$$(a)\ \begin{pmatrix} -3 & 0 \\ -4 & -1 \end{pmatrix}, \quad (b)\ \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}, \quad (c)\ \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad (d)\ \begin{pmatrix} -1 & 3 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & -1 \end{pmatrix},$$

$$(e)\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}, \quad (f)\ \begin{pmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad (g)\ \begin{pmatrix} 1 & 0 & -3 & -2 \\ -\frac{1}{2} & \frac{1}{2} & 2 & \frac{3}{2} \\ -\frac{1}{6} & 0 & \frac{3}{2} & \frac{2}{3} \\ \frac{2}{3} & 0 & -3 & -\frac{5}{3} \end{pmatrix}.$$

Solution:

(a) Unstable — eigenvalues $-1, -3$;
(b) unstable — eigenvalues $\frac{5+\sqrt{73}}{12} \approx 1.12867$, $\frac{5-\sqrt{73}}{12} \approx -.29533$;
(c) asymptotically stable — eigenvalues $\frac{1 \pm \mathrm{i}}{2}$;
(d) stable — eigenvalues $-1, \pm\,\mathrm{i}$;
(e) unstable — eigenvalues $\frac{5}{4}, \frac{1}{4}, \frac{1}{4}$;
(f) unstable — eigenvalues $2, 1, 1$;,
(g) asymptotically stable — eigenvalues $\frac{1}{2}, \frac{1}{3}, 0$.

10.2.4. (a) Determine the eigenvalues and spectral radius of the matrix $T = \begin{pmatrix} 3 & 2 & -2 \\ -2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$.

(b) Use this information to find the eigenvalues and spectral radius of $\widehat{T} = \begin{pmatrix} \frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$.

(c) Write down an asymptotic formula for the solutions to the iterative system $\mathbf{u}^{(k+1)} = \widehat{T}\,\mathbf{u}^{(k)}$.

Solution:

(a) $\lambda_1 = 3$, $\lambda_2 = 1 + 2\,\mathrm{i}$, $\lambda_3 = 1 - 2\,\mathrm{i}$, $\rho(T) = 3$.
(b) $\lambda_1 = \frac{3}{5}$, $\lambda_2 = \frac{1}{5} + \frac{2}{5}\,\mathrm{i}$, $\lambda_3 = \frac{1}{5} - \frac{2}{5}\,\mathrm{i}$, $\rho(\widetilde{T}) = \frac{3}{5}$.
(c) $\mathbf{u}^{(k)} \approx c_1 \left(\frac{3}{5}\right)^k (-1, -1, 1)^T$, provided the initial data has a non-zero component, $c_1 \neq 0$, in the direction of the dominant eigenvector $(-1, -1, 1)^T$.

10.2.5. (a) Show that the spectral radius of $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is $\rho(T) = 1$. (b) Show that most iterates $\mathbf{u}^{(k)} = T^k \mathbf{u}^{(0)}$ become unbounded as $k \to \infty$. (c) Discuss why the inequality $\| \mathbf{u}^{(k)} \| \leq C\,\rho(T)^k$ does not hold when the coefficient matrix is incomplete. (d) Can you prove that (10.28) holds in this example?

Solution:

(*a*) T has a double eigenvalue of 1, so $\rho(T) = 1$.

(*b*) Set $\mathbf{u}^{(0)} = \begin{pmatrix} a \\ b \end{pmatrix}$. Then $T^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, and so $\mathbf{u}^{(k)} = \begin{pmatrix} a + k\,b \\ b \end{pmatrix} \to \infty$ provided $b \neq 0$.

(*c*) In this example, $\|\,\mathbf{u}^{(k)}\,\| = \sqrt{b^2\,k^2 + 2\,a\,b\,k + a^2 + b^2} \approx b\,k \to \infty$ when $b \neq 0$, while $C\,\rho(T)^k = C$ is constant, so eventually $\|\,\mathbf{u}^{(k)}\,\| > C\,\rho(T)^k$ no matter how large C is.

(*d*) For any $\sigma > 1$, we have $b\,k \leq C\,\sigma^k$ for $k \geq 0$ provided $C \gg 0$ is sufficiently large — more specifically, if $C > b/(e \log \sigma)$; see Exercise 10.2.22.

10.2.6. Given a linear iterative system with non-convergent matrix, which solutions, if any, will converge to $\mathbf{0}$?

Solution: A solution $\mathbf{u}^{(k)} \to \mathbf{0}$ if and only if the initial vector $\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + \cdots + c_j\,\mathbf{v}_j$ is a linear combination of the eigenvectors (or more generally, Jordan chain vectors) corresponding to eigenvalues satisfying $|\,\lambda_i\,| < 1$, $i = 1, \ldots, j$.

10.2.7. Prove that if A is any square matrix, then there exists $c \neq 0$ such that the scalar multiple $c\,A$ is a convergent matrix. Find a formula for the largest possible such c.

Solution: Since $\rho(c\,A) = |\,c\,|\,\rho(A)$, then $c\,A$ is convergent if and only if $|\,c\,| < 1/\rho(A)$. So, technically, there isn't a largest c.

♢ 10.2.8. Suppose T is a complete matrix. (*a*) Prove that every solution to the corresponding linear iterative system is bounded if and only if $\rho(T) \leq 1$. (*b*) Can you generalize this result to incomplete matrices? *Hint*: Look at Exercise 10.1.36.

Solution:

(a) Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be a unit eigenvector basis for T, so $\|\,\mathbf{u}_j\,\| = 1$. Let

$$m_j = \max\left\{\, |\,c_j\,| \;\Big|\; \|\,c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n\,\| \leq 1 \,\right\},$$

which is finite since we are maximizing a continuous function over a closed, bounded set. Set $m^\star = \max\{m_1, \ldots, m_n\}$. Now, if

$$\|\,\mathbf{u}^{(0)}\,\| = \|\,c_1\,\mathbf{u}_1 + \cdots + c_n\,\mathbf{u}_n\,\| < \varepsilon, \qquad \text{then} \qquad |\,c_j\,| < m^\star \varepsilon \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore, by (10.25),

$$\|\,\mathbf{u}^{(k)}\,\| = \leq |\,c_1\,| + \cdots + |\,c_n\,| \leq n\,m^\star \varepsilon,$$

and hence the solution remains close to $\mathbf{0}$. *Q.E.D.*

(*b*) If any eigenvalue of modulus $\|\,\lambda\,\| = 1$ is incomplete, then, according to (10.23), the system has solutions of the form $\mathbf{u}^{(k)} = \lambda^k\,\mathbf{w}_i + k\,\lambda^{k-1}\,\mathbf{w}_{i-1} + \cdots$, which are unbounded as $k \to \infty$. Thus, the origin is not stable in this case. On the other hand, if all avs of modulus 1 are complete, then the system is stable, even if there are incomplete eigenvalues of modulus < 1.

10.2.9. Suppose a convergent iterative system has a single dominant real eigenvalue λ_1. Discuss how the asymptotic behavior of the real solutions depends on the sign of λ_1.

Solution: Assume $\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + \cdots + c_n\,\mathbf{v}_n$ with $c_1 \neq 0$. For $k \gg 0$, $\mathbf{u}^{(k)} \approx c_1\,\lambda_1^k\,\mathbf{v}_1$ since $|\,\lambda_1^k\,| \gg |\,\lambda_j^k\,|$ for all $j > 1$. Thus, the entries satisfy $u_i^{(k+1)} \approx \lambda_1\,u_i^{(k)}$ and so, if nonzero, are just multiplied by λ_1. Thus, if $\lambda_1 > 0$ we expect to see the signs of all the entries of $\mathbf{u}^{(k)}$ not change for k sufficiently large, whereas if $\lambda_1 < 0$, the signs alterate at each step of the iteration.

♡ 10.2.10. (a) Discuss the asymptotic behavior of solutions to a convergent iterative system that has two eigenvalues of largest modulus, e.g., $\lambda_1 = -\lambda_2$. How can you detect this? (*b*) Discuss the case when A is a real matrix with a complex conjugate pair of dominant eigenvalues.

Solution: Writing $\mathbf{u}^{(0)} = c_1\,\mathbf{v}_1 + \;\cdots\; + c_n\,\mathbf{v}_n$, then for $k \gg 0$, $\mathbf{u}^{(k)} \approx \lambda_1^k\,(c_1\,\mathbf{v}_1 + c_2\,(\lambda_1/\lambda_2)^k\,\mathbf{v}_2)$. T ■

10.2.11. Suppose T has spectral radius $\rho(T)$. Can you predict the spectral radius of $c\,T + d\,\mathrm{I}$, where c, d are scalars? If not, what additional information do you need?

Solution: If T has eigenvalues λ_j, then $c\,T + d\,\mathrm{I}$ has eigenvalues $c\,\lambda_j + d$. However, it is not necessarily true that the dominant eigenvalue of $c\,T + d\,\mathrm{I}$ is $c\,\lambda_1 + d$ when λ_1 is the dominant eigenvalue of T. For instance, if $\lambda_1 = 3, \lambda_2 = -2$, so $\rho(T) = 3$, then $\lambda_1 - 2 = 1, \lambda_2 = -4$, so $\rho(T - 2\,\mathrm{I}\,) = 4 \neq \rho(T) - 2$. Thus, you need to know all the eigenvalues to predict $\rho(T)$. (In more detail, it actually suffices to know the extreme eigenvalues, i.e., those such that the other eigenvalues lie in their convex hull in the complex plane.)

10.2.12. Let A have singular values $\sigma_1 \geq \cdots \geq \sigma_n$. Prove that $A^T A$ is a convergent matrix if and only if $\sigma_1 < 1$. (Later we will show that this implies that A itself is convergent.)

Solution: By definition, the eigenvalues of $A^T A$ are $\lambda_i = \sigma_i^2$, and so the spectral radius of $A^T A$ is equal to $\rho(A^T A) = \max\{\,\sigma_1^2, \ldots, \sigma_n^2\,\}$. Thus $\rho(A^T A) = \lambda_1 < 1$ if and only if $\sigma_1 = \sqrt{\lambda_1} < 1$.

♡ 10.2.13. Let M_n be the $n \times n$ tridiagonal matrix with all 1's on the sub- and super-diagonals, and zeros on the main diagonal. (*a*) What is the spectral radius of M_n? *Hint*: Use Exercise 8.2.46. (*b*) Is M_n convergent? (*c*) Find the general solution to the iterative system $\mathbf{u}^{(k+1)} = M_n\,\mathbf{u}^{(k)}$.

Solution: (*a*) $\rho(M_n) = 2\cos\frac{\pi}{n+1}$. (*b*) No since its spectral radius is slightly less than 2. (*c*) $\mathbf{u}_i^{(k)} = \displaystyle\sum_{j=1}^{n} c_j\left(2\cos\frac{j\,\pi}{n+1}\right)^k \sin\frac{i\,j\,\pi}{n+1}, \quad i = 1, \ldots, n.$

♡ 10.2.14. Let α, β be scalars. Let $T_{\alpha,\beta}$ be the $n \times n$ tridiagonal matrix that has all α's on the sub- and super-diagonals, and β's on the main diagonal. (*a*) Solve the iterative system $\mathbf{u}^{(k+1)} = T_{\alpha,\beta}\,\mathbf{u}^{(k)}$. (*b*) For which values of α, β is the system asymptotically stable? *Hint*: Combine Exercises 10.2.13 and 10.1.30.

Solution:

(*a*) $\mathbf{u}_i^{(k)} = \displaystyle\sum_{j=1}^{n} c_j\left(\beta + 2\,\alpha\cos\frac{j\,\pi}{n+1}\right)^k \sin\frac{i\,j\,\pi}{n+1}, \quad i = 1, \ldots, n.$

(*b*) Stable if and only if $\rho(T_{\alpha,\beta}) = (\left|\,\beta \pm 2\,\alpha\cos\dfrac{\pi}{n+1}\,\right| < 1$. In particular, if $|\,\beta \pm 2\,\alpha\,| < 1$ the system is asymptotically stable for any n.

10.2.15. (*a*) Prove that if $|\det T\,| > 1$ then the iterative system $\mathbf{u}^{(k+1)} = T\,\mathbf{u}^{(k)}$ is unstable. (*b*) If $|\det T\,| < 1$ is the system necessarily asymptotically stable? Prove or give a counterexample.

Solution:

(*a*) According to Exercise 8.2.24, T has at least one eigenvalue with $|\,\lambda\,| > 1$.

(*b*) No. For example $T = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ has $\det T = \frac{2}{3}$ but $\rho(T) = 2$.

10.2.16. *True or false*: (*a*) $\rho(c\,A) = c\,\rho(A)$, (*b*) $\rho(S^{-1} A\,S) = \rho(A)$, (*c*) $\rho(A^2) = \rho(A)^2$, (*d*) $\rho(A^{-1}) = 1/\rho(A)$, (*e*) $\rho(A + B) = \rho(A) + \rho(B)$, (*f*) $\rho(A\,B) = \rho(A)\,\rho(B)$.

Solution: (*a*) False: $\rho(c\,A) = |\,c\,|\,\rho(A)$. (*b*) True. (*c*) True. (*d*) False since $\rho(A) = \max\lambda$ whereas $\rho(A^{-1}) = \max 1/\lambda$. (*e*) False in almost all cases. (*f*) False.

10.2.17. *True or false*: (*a*) If A is convergent, then A^2 is convergent. (*b*) If A is convergent, then $A^T A$ is convergent.

Solution: (a) True by part (*c*) of Exercise 10.2.16. (*b*) False. $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$ has $\rho(A) = \frac{1}{2}$ whereas $A^T A = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{4} \end{pmatrix}$ has $\rho(A^T A) = \frac{3}{4} + \frac{1}{2}\sqrt{2} = 1.45711$.

10.2.18. *True or false*: If the zero solution of the differential equation $\dot{\mathbf{u}} = A\mathbf{u}$ is asymptotically stable, so is the zero solution of the iterative system $\mathbf{u}^{(k+1)} = A\mathbf{u}^{(k)}$.

Solution: False. The first requires $\operatorname{Re}\lambda_j < 0$; the second requires $|\lambda_j| < 1$.

10.2.19. Suppose $T^k \to A$ as $k \to \infty$. (*a*) Prove that $A^2 = A$. (*b*) Can you characterize all such matrices A? (*c*) What are the conditions on the matrix T for this to happen?

Solution: (a) $A^2 = \left(\lim_{k\to\infty} T^k \right)^2 = \lim_{k\to\infty} T^{2k} = A$. (*b*) The only eigenvalues of A are 1 and 0. Moreover, A must be complete, since if $\mathbf{v}_1, \mathbf{v}_2$ are the first two vectors in a Jordan chain, then $A\mathbf{v}_1 = \lambda\mathbf{v}_1$, $A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, with $\lambda = 0$ or 1, but $A^2\mathbf{v}_2 = \lambda^2\mathbf{v}_1 + 2\lambda\mathbf{v}_2 \neq A\mathbf{v}_2 = \lambda\mathbf{v}_2 + \mathbf{v}_1$, so there are no Jordan chains except for the ordinary eigenvectors. Therefore, $A = S\operatorname{diag}(1, \ldots, 1, 0, \ldots 0)\, S^{-1}$ for some nonsingular matrix S. (*c*) If λ is an eigenvalue of T, then either $|\lambda| < 1$, or $\lambda = 1$ and is a complete eigenvalue.

10.2.20. Prove that a matrix with all integer entries is convergent if and only if it is nilpotent, i.e., $A^k = \mathrm{O}$ for some k. Give a nonzero example of such a matrix.

Solution: If $\mathbf{v}$ has integer entries, so does $A^k\mathbf{v}$ for any k, and so the only way in which $A^k\mathbf{v} \to \mathbf{0}$ is if $A^k\mathbf{v} = \mathbf{0}$ for some k. Now consider the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Let k_i be such that $A^{k_i}\mathbf{e}_i = \mathbf{0}$. Let $k = \max\{k_1, \ldots, k_n\}$, so $A^k\mathbf{e}_i = \mathbf{0}$ for all $i = 1, \ldots, n$. Then $A^k\,\mathrm{I} = A^k = \mathrm{O}$, and hence A is nilpotent. *Q.E.D.* The simplest example is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

♥ 10.2.21. Consider a second order iterative scheme $\mathbf{u}^{(k+2)} = A\mathbf{u}^{(k+1)} + B\mathbf{u}^{(k)}$. Define a *generalized eigenvalue* to be a complex number that satisfies $\det(\lambda^2\,\mathrm{I} - \lambda A - B) = 0$. Prove that the system is asymptotically stable if and only if all its generalized eigenvalues satisfy $|\lambda| < 1$. *Hint*: Look at the equivalent first order system and use Exercise 1.9.23b.

Solution: The equivalent first order system $\mathbf{v}^{(k+1)} = C\mathbf{v}^{(k)}$ for $\mathbf{v}^{(k)} = \begin{pmatrix} \mathbf{u}^{(k)} \\ \mathbf{u}^{(k+1)} \end{pmatrix}$ has coefficient matrix $C = \begin{pmatrix} \mathrm{O} & \mathrm{I} \\ B & A \end{pmatrix}$. To compute the eigenvalues of C we form $\det(C - \lambda\,\mathrm{I}) = \det\begin{pmatrix} -\lambda\,\mathrm{I} & \mathrm{I} \\ B & A - \lambda\,\mathrm{I} \end{pmatrix}$. Now use row operations to subtract appropriate multiples of the first n rows from the last n, and the a series of row interchanges to conclude that

$$\begin{aligned} \det(C - \lambda\,\mathrm{I}) &= \det\begin{pmatrix} -\lambda\,\mathrm{I} & \mathrm{I} \\ B + \lambda A - \lambda^2\,\mathrm{I} & \mathrm{O} \end{pmatrix} = \pm\det\begin{pmatrix} B + \lambda A - \lambda^2\,\mathrm{I} & \mathrm{O} \\ -\lambda\,\mathrm{I} & \mathrm{I} \end{pmatrix} \\ &= \pm\det(B + \lambda A - \lambda^2\,\mathrm{I}). \end{aligned}$$

Thus, the generalized eigenvalues are the same as the eigenvalues of C, and hence stability requires they all satisfy $|\lambda| < 1$.

♢ 10.2.22. Let $p(t)$ be a polynomial. Assume $0 < \lambda < \mu$. Prove that there is a positive constant C such that $p(n)\,\lambda^n < C\,\mu^n$ for all $n > 0$.

Solution: Set $\sigma = \mu/\lambda > 1$. If $p(x) = c_k x^k + \cdots + c_1 x + c_0$ has degree k, then $p(n) \le a\,n^k$ for all $n \ge 1$ where $a = \max |\,c_i\,|$. To prove $a\,n^k \le C\,\sigma^n$ it suffice to prove that $k \log n < n \log \sigma + \log C - \log a$. Now $h(n) = n \log \sigma - k \log n$ has a minimum when $h'(n) \log \sigma - k/n = 0$, so $n = k/\log \sigma$. The minimum value is $h(k/\log \sigma) = k(1 - \log(k/\log \sigma))$. Thus, choosing $\log C > \log a + k(\log(k/\log \sigma) - 1)$ will ensure the desired inequality. *Q.E.D.*

♢ 10.2.23. Prove the inequality (10.28) when T is incomplete. Use it to complete the proof of Theorem 10.14 in the incomplete case. *Hint*: Use Exercises 10.1.36, 10.2.22.

Solution: According to Exercise 10.1.36, there is a polynomial $p(x)$ such that

$$\|\,\mathbf{u}^{(k)}\,\| \le \sum_i |\,\lambda_i\,|^k p_i(k) \le p(k)\,\rho(A)^k.$$

Thus, by Exercise 10.2.22, $\|\,\mathbf{u}^{(k)}\,\| \le C\,\sigma^k$ for any $\sigma > \rho(A)$. *Q.E.D.*

♢ 10.2.24. Suppose that M is a nonsingular matrix. (*a*) Prove that the *implicit iterative scheme* $M\,\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)}$ is asymptotically stable if and only if all the eigenvalues of M are strictly greater than one in magnitude: $|\,\mu_i\,| > 1$. (*b*) Let K be another matrix. Prove that iterative scheme $M\,\mathbf{u}^{(n+1)} = K\,\mathbf{u}^{(n)}$ is asymptotically stable if and only if all the generalized eigenvalues of the matrix pair K, M, as in Exercise 8.4.8 are strictly less than one in magnitude: $|\,\lambda_i\,| < 1$.

Solution:

(*a*) Rewriting the system as $\mathbf{u}^{(n+1)} = M^{-1}\mathbf{u}^{(n)}$, stability requires $\rho(M^{-1}) < 1$. The eigenvalues of M^{-1} are the reciprocals of the eigenvalues of M, and hence $\rho(M^{-1}) < 1$ if and only if $1/|\,\mu_i\,| < 1$. *Q.E.D.*

(*b*) Rewriting the system as $\mathbf{u}^{(n+1)} = M^{-1}K\,\mathbf{u}^{(n)}$, stability requires $\rho(M^{-1}K) < 1$. Moreover, the eigenvalues of $M^{-1}K$ coincide with the generalized eigenvalues of the pair; see Exercise 9.5.32 for details.

10.2.25. Find all fixed points for the linear iterative systems with the following coefficient matrices: (*a*) $\begin{pmatrix} .7 & .3 \\ .2 & .8 \end{pmatrix}$, (*b*) $\begin{pmatrix} .6 & 1.0 \\ .3 & -.7 \end{pmatrix}$, (*c*) $\begin{pmatrix} -1 & -1 & -4 \\ -2 & 0 & -4 \\ 1 & -1 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 2 & 1 & -1 \\ 2 & 3 & -2 \\ -1 & -1 & 2 \end{pmatrix}$.

Solution: (*a*) all scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; (*b*) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$; (*c*) all scalar multiples of $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$;

(*d*) all linear combinations of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

10.2.26. (*a*) Discuss the stability of each fixed point and the asymptotic behavior(s) of the solutions to the systems in Exercise 10.2.25. (*b*) Which fixed point, if any, does the solution with initial condition $\mathbf{u}^{(0)} = \mathbf{e}_1$ converge to?

Solution:

(*a*) The eigenvalues are $1, \frac{1}{2}$, so the fixed points are stable, while all other solutions go to a unique fixed point at rate $\left(\frac{1}{2}\right)^k$. When $\mathbf{u}^{(0)} = (1, 0)^T$, then $\mathbf{u}^{(k)} \to \left(\frac{3}{5}, \frac{3}{5}\right)^T$.

(b) The eigenvalues are $-.9, .8$, so the origin is a stable fixed point, and every nonzero solution goes to it, most at a rate of $.9^k$. When $\mathbf{u}^{(0)} = (1, 0)^T$, then $\mathbf{u}^{(k)} \to \mathbf{0}$ also.

(c) The eigenvalues are $-2, 1, 0$, so the fixed points are unstable. Most solutions, specifically those with a nonzero component in the dominant eigenvector direction, become unbounded. However, when $\mathbf{u}^{(0)} = (1, 0, 0)^T$, then $\mathbf{u}^{(k)} = (-1, -2, 1)^T$ for $k \geq 1$, and the solution stays at a fixed point.

(d) The eigenvalues are 5 and 1, so the fixed points are unstable. Most solutions, specifically those with a nonzero component in the dominant eigenvector direction, become unbounded, including that with $\mathbf{u}^{(0)} = (1, 0, 0)^T$.

10.2.27. Suppose T is a symmetric matrix that satisfies the hypotheses of Theorem 10.17 with a simple eigenvalue $\lambda_1 = 1$. Prove the solution to the linear iterative system has limiting value $\displaystyle \lim_{k \to \infty} \mathbf{u}^{(k)} = \frac{\mathbf{u}^{(0)} \cdot \mathbf{v}_1}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1$.

Solution: Since T is symmetric, its eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthonormal basis of $\mathbb{R}^n$. Writing $\mathbf{u}^{(0)} = c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$, the coefficients are given by the usual orthogonality formula (5.7): $c_i = \mathbf{u}^{(0)} \cdot \mathbf{v}_i / \| \mathbf{v}_1 \|^2$. Moreover, since $\lambda_1 = 1$, while $| \lambda_j | < 1$ for $j \geq 2$,

$$\mathbf{u}^{(k)} = c_1 \mathbf{v}_1 + c_2 \lambda_2^k \mathbf{v}_2 + \cdots + c_n \lambda_n^k \mathbf{v}_n \longrightarrow c_1 \mathbf{v}_1 = \frac{\mathbf{u}^{(0)} \cdot \mathbf{v}_1}{\| \mathbf{v}_1 \|^2} \mathbf{v}_1. \qquad Q.E.D.$$

10.2.28. *True or false*: If T has a stable nonzero fixed point, then it is a convergent matrix.

Solution: False — T has an eigenvalue of 1, but convergence requires all eigenvalues less than 1 in modulus.

10.2.29. *True or false*: If every point $\mathbf{u} \in \mathbb{R}^n$ is a fixed point, then they are all stable. Characterize such systems.

Solution: True. In this case $T = \mathrm{I}$ and all solutions remain fixed.

♡ 10.2.30. (a) Under what conditions does the linear iterative system $\mathbf{u}^{(k+1)} = T \mathbf{u}^{(k)}$ have a period 2 solution, i.e., $\mathbf{u}^{(k+2)} = \mathbf{u}^{(k)} \neq \mathbf{u}^{(k+1)}$? Give an example of such a system. (b) Under what conditions is there a unique period 2 solution? (c) What about a period m solution?

Solution:

(a) The condition $\mathbf{u}^{(k+2)} = T^2 \mathbf{u}^{(k)} = \mathbf{u}^{(k)}$ implies that $\mathbf{u}^{(k)} \neq \mathbf{0}$ is an eigenvector of T^2 with eigenvalue of 1. Thus, $\mathbf{u}^{(k)}$ is an eigenvector of T with eigenvalue -1; if the eigenvalue were 1 then $\mathbf{u}^{(k)} = \mathbf{u}^{(k+1)}$, contrary to assumption. Thus, the iterative system has a period 2 solution if and only if T has an eigenvalue of -1.

(b) $T = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$ has the period 2 orbit $\mathbf{u}^{(k)} = \begin{pmatrix} c(-1)^k \\ 0 \end{pmatrix}$ for any c.

(c) The period 2 solution is never unique since any nonzero scalar multiple is also a period 2 solution.

(d) T must have an eigenvalue equal to a primitive m^{th} root of unity.

◇ 10.2.31. Prove Theorem 10.18, (a) assuming T is complete; (b) for general T. *Hint*: Use Exercise 10.1.36.

Solution:

(a) ■

mnorm **10.3. Matrix Norms and the Gerschgorin Theorem.**

10.3.1. Compute the ∞ matrix norm of the following matrices. Which are guaranteed to be convergent? (a) $\begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} \end{pmatrix}$, (b) $\begin{pmatrix} \frac{5}{3} & \frac{4}{3} \\ -\frac{7}{6} & -\frac{5}{6} \end{pmatrix}$, (c) $\begin{pmatrix} \frac{2}{7} & -\frac{2}{7} \\ -\frac{2}{7} & \frac{6}{7} \end{pmatrix}$, (d) $\begin{pmatrix} \frac{1}{4} & \frac{3}{2} \\ -\frac{1}{2} & \frac{5}{4} \end{pmatrix}$,

(e) $\begin{pmatrix} \frac{2}{7} & \frac{2}{7} & -\frac{4}{7} \\ 0 & \frac{2}{7} & \frac{6}{7} \\ \frac{2}{7} & \frac{4}{7} & \frac{2}{7} \end{pmatrix}$, (f) $\begin{pmatrix} 0 & .1 & .8 \\ -.1 & 0 & .1 \\ -.8 & -.1 & 0 \end{pmatrix}$, (g) $\begin{pmatrix} 1 & -fr23 & -fr23 \\ 1 & -\frac{1}{3} & -1 \\ \frac{1}{3} & -\frac{2}{3} & 0 \end{pmatrix}$, $\begin{pmatrix} \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$.

Solution: (a) $\frac{3}{4}$, convergent. (b) 3, inconclusive. (c) $\frac{8}{7}$, inconclusive. (d) $\frac{7}{4}$, inconclusive. (e) $\frac{8}{7}$, inconclusive. (f) .9, convergent. (g) $\frac{7}{3}$, inconclusive. (h) 1, inconclusive.

10.3.2. Compute the Euclidean matrix norm of each matrix in Exercise 10.3.1. Have your convergence conclusions changed?

Solution: (a) .671855, convergent. (b) 2.5704, inconclusive. (c) .9755, convergent. (d) 1.9571, inconclusive. (e) 1.1066, inconclusive. (f) .8124, convergent. (g) 2.03426, inconclusive. (h) .7691, convergent.

10.3.3. Compute the spectral radii of the matrices in Exercise 10.3.1. Which are convergent? Compare your conclusions with those of Exercises 10.3.1 and 2.

Solution: (a) $\frac{2}{3}$, convergent. (b) $\frac{1}{2}$, convergent. (c) .9755, convergent. (d) 1.0308, divergent. (e) .9437, convergent. (f) .8124, convergent. (g) $\frac{2}{3}$, convergent. (h) $\frac{2}{3}$, convergent.

10.3.4. Let k be an integer and set $A_k = \begin{pmatrix} k & -1 \\ k^2 & -k \end{pmatrix}$. Compute (a) $\| A_k \|_\infty$, (b) $\| A_k \|_2$, (c) $\rho(A_k)$. (d) Explain why every A_k is a convergent matrix, even though their matrix norms can be arbitrarily large. (e) Why does this not contradict Corollary 10.32?

Solution: (a) $\| A_k \|_\infty = k^2 + k$, (b) $\| A_k \|_2 = k^2 + 1$, (c) $\rho(A_k) = 0$. (d) Thus, a convergent matrix can have arbitrarily large norm. (e) Because the norm will depend on k.

10.3.5. Find a matrix A such that (a) $\| A^2 \|_\infty \neq \| A \|_\infty^2$; (b) $\| A^2 \|_2 \neq \| A \|_2^2$.

Solution: When $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, so $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then

(a) $\| A \|_\infty = 2$, $\| A^2 \|_\infty = 3$. (b) $\| A \|_2 = \sqrt{\frac{3+\sqrt{5}}{2}} = 1.6180$, $\| A^2 \|_2 = \sqrt{3 + 2\sqrt{2}} = 2.4142$.

10.3.6. Show that if $| c | < 1/\| A \|$ then $c A$ is a convergent matrix.

Solution: Since $\| c A \| = | c | \, \| A \| < 1$.

♢ 10.3.7. Prove that the spectral radius function does *not* satisfy the triangle inequality by finding matrices A, B such that $\rho(A + B) > \rho(A) + \rho(B)$.

Solution: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $\rho(A + B) = \sqrt{2} > 0 + 1 = \rho(A) + \rho(B)$.

10.3.8. *True or false*: If all singular values of A satisfy $\sigma_i < 1$ then A is convergent.

Solution: True — this implies $\| A \|_2 = \max \sigma_i < 1$.

10.3.9. Find a convergent matrix that has dominant singular value $\sigma_1 > 1$.

Solution: If $A = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$, then $\rho(A) = \frac{1}{2}$. The singular values of A are $\sigma_1 = \frac{\sqrt{3+2\sqrt{2}}}{2} = 1.2071$ and $\sigma_2 = \frac{\sqrt{3-2\sqrt{2}}}{2} = .2071$.

10.3.10. *True or false*: If $B = S^{-1} A S$ are similar matrices, then (*a*) $\| B \|_\infty = \| A \|_\infty$, (*b*) $\| B \|_2 = \| A \|_2$, (*c*) $\rho(B) = \rho(A)$.

Solution: (a) False: if $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, then $B = S^{-1} A S = \begin{pmatrix} 0 & -2 \\ 0 & 1 \end{pmatrix}$ and $\| B \|_\infty = 2 \neq 1 = \| A \|_\infty$; (*b*) false: same example has $\| B \|_2 = \sqrt{5} \neq \sqrt{2} = \| A \|_2$; (*c*) true, since A and B have the same eigenvalues.

10.3.11. Prove that the condition number of a nonsingular matrix is given by
$$\kappa(A) = \| A \|_2 \, \| A^{-1} \|_2.$$

Solution: By definition, $\kappa(A) = \sigma_1/\sigma_n$. Now $\| A \|_2 = \sigma_1$. On the other hand, by Exercise 8.5.12, the singular values of A^{-1} are the reciprocals $1/\sigma_i$ of the singular values of A, and so the largest one is $\| A^{-1} \|_2 = 1/\sigma_n$. *Q.E.D.*

♦ 10.3.12. (*i*) Find an explicit formula for the 1 matrix norm $\| A \|_1$. (*ii*) Compute the 1 matrix norm of the matrices in Exercise 10.3.1, and discuss convergence.

Solution: (*i*) The 1 matrix norm is the maximum absolute column sum. (*ii*) (*a*) $\frac{5}{6}$, convergent. (*b*) $\frac{17}{6}$, inconclusive. (*c*) $\frac{8}{7}$, inconclusive. (*d*) $\frac{11}{4}$, inconclusive. (*e*) $\frac{12}{7}$, inconclusive. (*f*) .9, convergent. (*g*) $\frac{7}{3}$, inconclusive. (*h*) $\frac{2}{3}$, convergent.

10.3.13. Prove directly from the axioms of Definition 3.12 that (10.40) defines a norm on the space of $n \times n$ matrices.

Solution: If $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are the rows of A, then the formula (10.40) can be rewritten as $\| A \|_\infty = \max\{ \| \mathbf{a}_i \|_1 \}$, i.e., the maximal 1 norm of the rows. Thus, by the properties of the 1-norm,
$$\begin{aligned} \| A + B \|_\infty &= \max\{ \| \mathbf{a}_i + \mathbf{b}_i \|_1 \} \le \max\{ \| \mathbf{a}_i \|_1 + \| \mathbf{b}_i \|_1 \} \\ &\le \max\{ \| \mathbf{a}_i \|_1 \} + \max\{ \| \mathbf{b}_i \|_1 \} = \| A \|_\infty + \| A \|_\infty, \\ \| c A \|_\infty &= \max\{ \| c\,\mathbf{a}_i \|_1 \} = \max\{ | c | \, \| \mathbf{a}_i \|_1 \} = | c | \, \max\{ \| \mathbf{a}_i \|_1 \} = | c | \, \| A \|_\infty. \end{aligned}$$
Finally, $\| A \|_\infty \ge 0$ since we are maximizing non-negative quantities; moreover, $\| A \|_\infty = 0$ if and only if all its rows have $\| \mathbf{a}_i \|_1 = 0$ and hence all $\mathbf{a}_i = \mathbf{0}$, which means $A = \mathrm{O}$. *Q.E.D.*

♦ 10.3.14. Let $K > 0$ be a positive definite matrix. Characterize the matrix norm induced by the inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T K \mathbf{y}$. *Hint*: Use Exercise 8.4.44.

Solution: $\| A \| = \max\{ \sigma_1, \ldots, \sigma_n \}$ is the largest generalized singular value, meaning $\sigma_i = \sqrt{\lambda_i}$ where $\lambda_1, \ldots, \lambda_n$ are the generalized eigenvalues of the positive definite matrix pair $A^T K A$ and K, satisfying $A^T K A \mathbf{v}_i = \lambda K \mathbf{v}_i$, or, equivalently, the eigenvalues of $K^{-1} A^T K A$.

10.3.15. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}$. Compute the matrix norm $\| A \|$ using the following norms in $\mathbb{R}^2$: (*a*) the weighted ∞ norm $\| \mathbf{v} \| = \max\{ 2 | v_1 |, 3 | v_2 | \}$; (*b*) the weighted 1 norm $\| \mathbf{v} \| = 2 | v_1 | + 3 | v_2 |$; (*c*) the weighted inner product norm $\| \mathbf{v} \| = \sqrt{2 v_1^2 + 3 v_2^2}$; (*d*) the norm associated with the positive definite matrix $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Solution:
(a) ■

♡ 10.3.16. The *Frobenius norm* of an $n \times n$ matrix A is defined as $\| A \|_F = \sqrt{\sum_{i,j=1}^{n} a_{ij}^2}$. Prove that this defines a matrix norm by checking the three norm axioms plus the multiplicative inequality (10.33).

Solution: If we identify an $n \times n$ matrix with a vector in $\mathbb{R}^{n^2}$, then the Frobenius norm is the same as the ordinary Euclidean norm, and so the norm axioms are immediate. To check the multiplicative property, let $\mathbf{r}_1^T, \ldots, \mathbf{r}_n^T$ denote the rows of A and $\mathbf{c}_1, \ldots, \mathbf{c}_n$ the columns of B, so $\| A \|_F = \sqrt{\sum_{i=1}^{n} \| \mathbf{r}_i \|^2}$, $\| B \|_F = \sqrt{\sum_{j=1}^{n} \| \mathbf{c}_j \|^2}$. Then, setting $C = AB$, we have

$$\| C \|_F = \sqrt{\sum_{i,j=1}^{n} c_{ij}^2} = \sqrt{\sum_{i,j=1}^{n} (\mathbf{r}_i^T \mathbf{c}_j)^2} \leq \sqrt{\sum_{i,j=1}^{n} \| \mathbf{r}_i \|^2 \| \mathbf{c}_j \|^2} = \| A \|_F \; \| B \|_F,$$

where we used Cauchy–Schwarz for the inequality. *Q.E.D.*

10.3.17. Let A be an $n \times n$ matrix with singular value vector $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_r)$. Prove that (a) $\| \boldsymbol{\sigma} \|_\infty = \| A \|_2$; (b) $\| \boldsymbol{\sigma} \|_2 = \| A \|_F$, the Frobenius norm of Exercise 10.3.16.
Remark: $\| \boldsymbol{\sigma} \|_1$ also defines a useful matrix norm, known as the *Ky Fan norm*.

Solution:
(a) This is a restatement of Proposition 10.28.
(b) $\| \boldsymbol{\sigma} \|_2^2 = \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \lambda_i = \operatorname{tr}(A^T A) = \sum_{i,j=1}^{n} a_{ij}^2 = \| A \|_F^2$.

10.3.18. Explain why $\| A \| = \max | a_{ij} |$ defines a norm on the space of $n \times n$ matrices. Show by example that this is *not* a matrix norm, i.e., (10.33) is not necessarily valid.

Solution: If we identify a matrix A with a vector in $\mathbb{R}^{n^2}$, then this agrees with the ∞ norm on $\mathbb{R}^{n^2}$ and hence satisfies the norm axioms. When $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, then $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and so $\| A^2 \| = 2 > 1 = \| A \|^2$.

10.3.19. Prove that the closed curve parametrized in (10.37) is an ellipse. What are its semi-axes?

Solution: ■

♢ 10.3.20. If V is a finite-dimensional normed vector space, then it can be proved, [**63**], that a series $\sum_{n=1}^{\infty} \mathbf{v}_i$ converges to $\mathbf{v} \in V$ if and only if the series of norms converges: $\sum_{n=1}^{\infty} \| \mathbf{v}_i \| < \infty$. Use this fact to prove that the exponential matrix series (9.45) converges.

Solution: $e^{tA} = \sum_{n=0}^{\infty} \dfrac{t^n}{n!} A^n$ and the series of norms is $\sum_{n=0}^{\infty} \dfrac{| t |^n}{n!} \| A \|^n = e^{| t | \, \| A \|}$ is the standard scalar exponential series, which, by the ratio or root tests, [**3**], converges for all t. *Q.E.D.*

10.3.21. (a) Use Exercise 10.3.20 to prove that the geometric matrix series $\sum_{n=0}^{\infty} A^n$ converges whenever $\rho(A) < 1$. *Hint*: Apply Corollary 10.32. (b) Prove that the sum is $(\mathrm{I} - A)^{-1}$. How do you know $\mathrm{I} - A$ is invertible?

Solution: (a) Choosing a matrix norm such that $a = \| A \| < 1$, the norm series is bounded by a convergent geometric series:

$$\sum_{\| \|=0}^{\infty} A^n \leq \sum_{\| \|=0}^{\infty} A^n = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}\,.$$

Therefore, the matrix series converges. (b) Moreover,

$$(\mathrm{I} - A) \sum_{n=0}^{\infty} A^n = \sum_{n=0}^{\infty} A^n - \sum_{n=0}^{\infty} A^{n+1} = \mathrm{I}\,,$$

since all other terms cancel. $\mathrm{I} - A$ is invertible if and only if 1 is not an eigenvalue of A, and we are assuming all eigenvalues are less than 1 in magnitude.

10.3.22. For each of the following matrices (i) Find all Gerschgorin disks; (ii) plot the Gerschgorin domain in the complex plane; (iii) compute the eigenvalues and confirm the truth of the Circle Theorem 10.34.

(a) $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$,

(e) $\begin{pmatrix} -1 & 3 & -3 \\ 2 & 2 & -7 \\ 0 & 3 & -4 \end{pmatrix}$, (f) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} & 0 \end{pmatrix}$, (g) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$, (h) $\begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}$.

Solution:

(a) Gerschgorin disks: $| z - 1 | \leq 2$; eigenvalues: $3, -1$.
(b) Gerschgorin disks: $| z - 1 | \leq \frac{2}{3}$, $\left| z + \frac{1}{6} \right| \leq \frac{1}{2}$; eigenvalues: $\frac{1}{2}, \frac{1}{3}$.
(c) Gerschgorin disks: $| z - 2 | \leq 3$, $| z | \leq 1$; eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$.
(d) Gerschgorin disks: $| z - 3 | \leq 1$, $| z - 2 | \leq 2$; eigenvalues: $4, 3, 1$.
(e) Gerschgorin disks: $| z + 1 | \leq 6$, $| z - 2 | \leq 9$, $| z + 4 | \leq 3$; eigenvalues: $-1, -1 \pm \mathrm{i}\sqrt{6}$.
(f) Gerschgorin disks: $z = \frac{1}{2}$, $| z | \leq \frac{1}{3}$, $| z | \leq \frac{5}{12}$; eigenvalues: $\frac{1}{2}, \pm \frac{1}{3\sqrt{2}}$.
(g) Gerschgorin disks: $| z | \leq 1$, $| z - 1 | \leq 1$; eigenvalues: $0, 1 \pm \mathrm{i}$.
(h) Gerschgorin disks: $| z - 3 | \leq 2$, $| z - 2 | \leq 1$, $| z | \leq 1$, $| z - 1 | \leq 2$;
eigenvalues: $\frac{1}{2} \pm \frac{\sqrt{5}}{2}, \frac{5}{2} \pm \frac{\sqrt{5}}{2}$.

10.3.23. *True or false*: The Gerschgorin domain of the transpose of a matrix A^T is the same as the Gerschgorin domain of the matrix A, that is $D_{A^T} = D_A$.

Solution: False. Almost any non-symmetric matrix, e.g., $\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ provides a counterexample.

♢ 10.3.24. (i) Explain why the eigenvalues of A must lie in its *refined Gerschgorin domain* $D_A^* = D_{A^T} \cap D_A$. (ii) Find the refined Gerschgorin domains for each of the matrices in Exercise 10.3.22 and confirm the result in part (i).

Solution:

(i) Because A and its transpose A^T have the same eigenvalues, which must therefore belong to both D_A and D_{A^T}.
(ii)

(a) Gerschgorin disks: $|z-1| \le 2$; eigenvalues: $3, -1$.
(b) Gerschgorin disks: $|z-1| \le \frac{1}{2}$, $\left| z + \frac{1}{6} \right| \le \frac{2}{3}$; eigenvalues: $\frac{1}{2}, \frac{1}{3}$.
(c) Gerschgorin disks: $|z-2| \le 1$, $|z| \le 3$; eigenvalues: $1 \pm \mathrm{i}\sqrt{2}$.
(d) Gerschgorin disks: $|z-3| \le 1$, $|z-2| \le 2$; eigenvalues: $4, 3, 1$.
(e) Gerschgorin disks: $|z+1| \le 2$, $|z-2| \le 6$, $|z+4| \le 10$; eigenvalues: $-1, -1 \pm \mathrm{i}\sqrt{6}$.
(f) Gerschgorin disks: $|z - \frac{1}{2}| \le \frac{1}{4}$, $|z| \le \frac{1}{6}$, $|z| \le \frac{1}{3}$; eigenvalues: $\frac{1}{2}, \pm \frac{1}{3\sqrt{2}}$.
(g) Gerschgorin disks: $z = 0$, $|z-1| \le 2$, $|z-1| \le 1$; eigenvalues: $0, 1 \pm \mathrm{i}$.
(h) Gerschgorin disks: $|z-3| \le 1$, $|z-2| \le 2$, $|z| \le 2$, $|z-1| \le 1$;
eigenvalues: $\frac{1}{2} \pm \frac{\sqrt{5}}{2}, \frac{5}{2} \pm \frac{\sqrt{5}}{2}$.

♢ 10.3.25. Let A be a square matrix. Prove that $max\{0,t\} \le \rho(A) \le s$, where $s = \max\{s_1, \ldots, s_n\}$ is the maximal absolute row sum of A, as defined in (10.39), and $t = \min\left\{ |a_{ii}| - r_i \right\}$, with r_i given by (10.44).

Solution: By elementary geometry, all points z in a closed disk of radius r centered at $z = a$ satisfy $\max\{0, |a| - r\} \le |z| \le |a| + r$. Thus, every point in the i^{th} Gerschgorin disk satisfies $\max\{0, |a_{ii}| - r_i\} \le |z| \le |a_{ii}| + r_i = s_i$. Since every eigenvalue lies in such a disk, they all satisfy $\max\{0,t\} \le |\lambda_i| \le s$, and hence $\rho(A) = \max\{|\lambda_i|\}$ does too. *Q.E.D.*

10.3.26. (a) Suppose that every entry of the $n \times n$ matrix A is bounded by $|a_{ij}| < \dfrac{1}{n}$. Prove that A is a convergent matrix. *Hint*: Use Exercise 10.3.25. (b) Produce a matrix of size $n \times n$ with one or more entries satisfying $|a_{ij}| = \dfrac{1}{n}$ that is not convergent.

Solution:

(a) The absolute row sums of A are bounded by $s_i = \displaystyle\sum_{j=1}^{n} |a_{ij}| < 1$, and so $\rho(A) \le s = \max s_i < 1$ by Exercise 10.3.25.

(b) $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ has eigenvalues $0, 1$ and hence $\rho(A) = 1$.

10.3.27. Suppose the largest entry (in modulus) of A is $|a_{ij}| = a_\star$. How large can its radius of convergence be?

Solution: Using Exercise 10.3.25, we find $\rho(A) \le s = \max\left\{ s_i \right\} \le n\, a_\star$.

10.3.28. Write down an example of a diagonally dominant matrix that is also convergent.

Solution: Any diagonal matrix whose diagonal entries satisfy $0 < |a_{ii}| < 1$.

10.3.29. *True or false*: (a) A positive definite matrix is diagonally dominant. (b) A diagonally dominant matrix is positive definite.

Solution: Both false. $\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ is a counterexample to (a), while $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a counterexample to (b). However, see Exercise 10.3.30.

♢ 10.3.30. Prove that if K is symmetric, diagonally dominant, and each diagonal entry is positive, then K is positive definite.

Solution: The eigenvalues of K are real by Theorem 8.20. The i^{th} Gerschgorin disk is centered at $k_{ii} > 0$ and by diagonal dominance its radius is less than the distance from its center to the origin. Therefore, all eigenvalues of K must be positive and hence, by Theorem 8.23, $K > 0$.

10.3.31. (*a*) Write down an invertible matrix A whose Gerschgorin domain contains 0. (*b*) Can you find an example which is also diagonally dominant?

Solution: (*a*) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has Gerschgorin domain $|\, z - 1 \,| \leq 2$. (*b*) No — see the proof of Proposition 10.37.

10.3.32. Prove that if A is diagonally dominant and each diagonal entry is negative, then the zero equilibrium solution to the linear system of ordinary differential equations $\dot{\mathbf{u}} = A\,\mathbf{u}$ is asymptotically stable.

Solution: The i^{th} Gerschgorin disk is centered at $a_{ii} < 0$ and, by diagonal dominance, its radius is less than the distance to the origin. Therefore, all eigenvalues of A lie in the left half plane: $\operatorname{Re} \lambda < 0$, which, by Theorem 9.15, implies asymptotic stability of the differential equation.

MP 10.4. Markov Processes.

10.4.1. Determine if the following matrices are regular transition matrices. If so, find the associated probability eigenvector. (*a*) $\begin{pmatrix} \frac12 & \frac13 \\ \frac34 & \frac23 \end{pmatrix}$, (*b*) $\begin{pmatrix} \frac14 & \frac34 \\ \frac23 & \frac13 \end{pmatrix}$, (*c*) $\begin{pmatrix} \frac14 & \frac23 \\ \frac34 & \frac13 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & \frac15 \\ 1 & \frac45 \end{pmatrix}$, (*e*) $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (*f*) $\begin{pmatrix} .3 & .5 & .2 \\ .3 & .2 & .5 \\ .4 & .3 & .3 \end{pmatrix}$, (*g*) $\begin{pmatrix} .1 & .5 & 0 \\ .1 & .2 & 1 \\ .8 & .3 & 0 \end{pmatrix}$, (*h*) $\begin{pmatrix} .1 & .5 & .4 \\ .4 & .1 & .3 \\ .3 & 0 & .7 \end{pmatrix}$, (*i*) $\begin{pmatrix} \frac12 & \frac12 & \frac13 \\ \frac12 & 0 & \frac13 \\ 0 & \frac12 & \frac13 \end{pmatrix}$, (*j*) $\begin{pmatrix} 0 & .2 & 0 & 1 \\ .5 & 0 & .3 & 0 \\ 0 & .8 & 0 & 0 \\ .5 & 0 & .7 & 0 \end{pmatrix}$, (*k*) $\begin{pmatrix} .1 & .2 & .3 & .4 \\ .2 & .5 & .3 & .1 \\ .3 & .3 & .1 & .3 \\ .4 & .1 & .3 & .2 \end{pmatrix}$, (*l*) $\begin{pmatrix} 0 & .6 & 0 & .4 \\ .5 & 0 & .3 & .1 \\ 0 & .4 & 0 & .5 \\ .5 & 0 & .7 & 0 \end{pmatrix}$, (*m*) $\begin{pmatrix} .1 & .3 & .7 & 0 \\ .1 & .2 & 0 & .8 \\ 0 & .5 & 0 & .2 \\ .8 & 0 & .3 & 0 \end{pmatrix}$.

Solution:
(*a*) Not a transition matrix;
(*b*) not a transition matrix;
(*c*) regular transition matrix: $\left(\frac{8}{17}, \frac{9}{17} \right)^T$;
(*d*) regular transition matrix: $\left(\frac16, \frac56 \right)^T$;
(*e*) not a regular transition matrix;
(*f*) regular transition matrix: $\left(\frac13, \frac13, \frac13 \right)^T$;
(*g*) regular transition matrix: $(\, .2415, .4348, .3237 \,)^T$;
(*h*) not a transition matrix;
(*i*) regular transition matrix: $\left(\frac{6}{13}, \frac{4}{13}, \frac{3}{13} \right)^T = (\, .4615, .3077, .2308 \,)^T$;
(*j*) not a regular transition matrix;
(*k*) not a transition matrix;
(*l*) regular transition matrix (A^4 has all positive entries): $\left(\frac{251}{1001}, \frac{225}{1001}, \frac{235}{1001}, \frac{290}{1001} \right)^T = (\, .250749, .224775, .234765, .28971 \,)^T$;
(*m*) regular transition matrix: $(\, .2509, .2914, .1977, .2600 \,)^T$.

10.4.2. A study has determined that, on average, the occupation of a boy depends on that of his father. If the father is a farmer, there is a 30% chance that the son will be a blue collar laborer, a 30% chance he will be a white collar professional, and a 40% chance he will also be a farmer. If the father is a laborer, there is a 30% chance that the son will also be one, a 60% chance he will be a professional, and a 10% chance he will be a farmer. If the father is a professional, there is a 70% chance that the son will also be one, a 25% chance he will be a laborer, and a 5% chance he will be a farmer. (*a*) What is the probability that the grandson of a farmer will also be a farmer? (*b*) In the long run, what proportion of the male population will be farmers?

Solution: (*a*) 20.5%; (*b*) 9.76% farmers, 26.83% laborers, 63.41% professionals

10.4.3. The population of an island is divided into city and country residents. Each year, 5% of the residents of the city move to the country and 15% of the residents of the country move to the city. In 2003, 35,000 people live in the city and 25,000 in the country. Assuming no growth in the population, how many people will live in the city and how many will live in the country between the years 2004 and 2008? What is the eventual population distribution of the island?

Solution: 2004: 37,000 city, 23,000 country. 2005: 38,600 city, 21,400 country. 2006: 39,880 city, 20,120 country. 2007: 40,904 city, 19.096 country. 2008: 41,723 city, 18,277 country. Eventual: 45,000 in the city and 15,000 in the country.

10.4.4. A student has the habit that if she doesn't study one night, she is 70% certain of studying the next night. Furthermore, the probability that she studies two nights in a row is 50%. How often does she study in the long run?

Solution: 58.33% of the nights.

10.4.5. A traveling salesman visits the three cities of Atlanta, Boston, and Chicago. The matrix $\begin{pmatrix} 0 & .5 & .5 \\ 1 & 0 & .5 \\ 0 & .5 & 0 \end{pmatrix}$ describes the transition probabilities of his trips. Describe his travels in words, and calculate how often he visits each city on average.

Solution: When in Atlanta he always goes to Boston; when in Boston he has a 50% probability of going to either Atlanta or Chicago; when in Chicago he has a 50% probability of going to either Atlanta or Boston. Note that the transition matrix is regular because A^4 has all positive entries. In the long run, his visits average: Atlanta: 33.33%, Boston: 44.44%, Chicago: 22.22%.

10.4.6. A business executive is managing three branches, labeled A, B and C, of a corporation. She never visits the same branch on consecutive days. If she visits branch A one day, she visits branch B the next day. If she visits either branch B or C that day, then the next day she is twice as likely to visit branch A as to visit branch B or C. Explain why the resulting transition matrix is regular. Which branch does she visit the most often in the long run?

Solution: The transition matrix is $\begin{pmatrix} 0 & \frac{2}{3} & \frac{2}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$. She visits branch A 40% of the time, branch B 45% and branch C: 15%.

10.4.7. A certain plant species has either red, pink or white flowers, depending on its genotype. If you cross a pink plant with any other plant, the probability distribution of the offspring are prescribed by the transition matrix $T = \begin{pmatrix} .5 & .25 & 0 \\ .5 & .5 & .5 \\ 0 & .25 & .5 \end{pmatrix}$. On average, if you continue only crossing with pink plants, what percentage of the three types of flowers would you expect to see in your garden?

Solution: 25% red, 50% pink, 25% pink.

10.4.8. Explain why the irregular Markov process with transition matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ does not reach a steady state. Use a population model to interpret what is going on.

Solution: If $\mathbf{u}^{(0)} = (a, b)^T$ is the initial state vector, then the subsequent state vectors switch back and forth between $(b, a)^T$ and $(a, b)^T$. At each step in the process, all of the population in state 1 goes to state 2 and vice versa, so the system never settles down.

10.4.9. A genetic model describing inbreeding, in which mating takes place only between individuals of the same genotype, is given by the Markov process $\mathbf{u}^{(n+1)} = T\,\mathbf{u}^{(n)}$, where $T = \begin{pmatrix} 1 & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 1 \end{pmatrix}$ is the transition matrix and $\mathbf{u}^{(n)} = \begin{pmatrix} p_n \\ q_n \\ r_n \end{pmatrix}$, whose entries are, respectively, the proportion of populations of genotype AA, Aa, aa in the n^{th} generation. Find the solution to this Markov process and analyze your result.

Solution: This is not a regular transition matrix, so we need to analyze the iterative process directly. The eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \frac{1}{2}$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Thus, the solution with initial value $\mathbf{u}^{(0)} = \begin{pmatrix} p_0 \\ q_0 \\ r_0 \end{pmatrix}$ is

$$\mathbf{u}^{(n)} = \left(p_0 + \tfrac{1}{2}\, q_0 \right) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left(\tfrac{1}{2}\, q_0 + r_0 \right) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{q_0}{2^{k+1}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} p_0 + \frac{1}{2}\, q_0 \\ 0 \\ \frac{1}{2}\, q_0 + r_0 \end{pmatrix}.$$

Therefore, this breeding process eventually results in a population with individuals of genotypes AA and aa only, the proportions of each depending upon the initial population.

◇ 10.4.10. Let T be a regular transition matrix with probability eigenvector $\mathbf{v}$. Prove that $\lim\limits_{k \to \infty} T^k = (\,\mathbf{v}\ \mathbf{v}\ \ldots\ \mathbf{v}\,)$ is a matrix with every column equal to $\mathbf{v}$.

Solution: The i^{th} column of T^k is $\mathbf{u}_i^{(k)} = T^k \mathbf{e}_i \to \mathbf{v}$ by Theorem 10.40. *Q.E.D.*

10.4.11. Find $\lim\limits_{k \to \infty} T^k$ when $T = \begin{pmatrix} .8 & .1 & .1 \\ .1 & .8 & .1 \\ .1 & .1 & .8 \end{pmatrix}$.

Solution: Use Exercise 10.4.10: $\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

10.4.12. Prove that, for all $0 \le p, q \le 1$ with $p + q > 0$, the probability eigenvector of the transition matrix $T = \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix}$ is $\mathbf{v} = \left(\dfrac{q}{p+q}, \dfrac{p}{p+q} \right)^T$.

Solution: First, $\mathbf{v}$ is a probability vector since the sum of its entries is $\dfrac{p}{p+q} + \dfrac{p}{p+q} = 1$. Moreover, $A\mathbf{v} = \left(\dfrac{(1-p)\,q + q\,p}{p+q} \quad \dfrac{p\,q + (1-q)\,p}{p+q} \right) = \left(\dfrac{q}{p+q} \quad \dfrac{p}{p+q} \right) = \mathbf{v}$, proving it is an eigenvector for eigenvalue 1.

10.4.13. A transition matrix is called *doubly stochastic* if both its row and column sums are equal to 1. What is the limiting probability state of a Markov chain with doubly stochastic transition matrix?

Solution: All equal probabilities: $\mathbf{z} = \left(\frac{1}{n}, \ldots, \frac{1}{n} \right)^T$.

10.4.14. Describe the final state of a Markov chain with symmetric transition matrix $T = T^T$.

Solution: $\mathbf{z} = \left(\frac{1}{n}, \ldots, \frac{1}{n} \right)^T$.

10.4.15. *True or false*: If T and T^T are both transition matrices, then $T = T^T$.

Solution: False: $\begin{pmatrix} .3 & .5 & .2 \\ .3 & .2 & .5 \\ .4 & .3 & .3 \end{pmatrix}$ is a counterexample.

10.4.16. *True or false*: If T is a transition matrix, so is T^{-1}.

Solution: False. For instance, if $T = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}$, then $T^{-1} = \begin{pmatrix} 4 & -2 \\ -3 & 3 \end{pmatrix}$, while $T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is not even invertible.

10.4.17. *True or false*: The set of all probability vectors forms a subspace of $\mathbb{R}^n$.

Solution: False. For instance, $\mathbf{0}$ is not a probability vector.

10.4.18. *Multiple Choice*: Every probability vector in $\mathbb{R}^n$ lies on the unit sphere for the (*a*) 1 norm, (*b*) 2 norm, (*c*) ∞ norm, (*d*) all of the above, (*e*) none of the above.

Solution: The 1 norm.

10.4.19. *True or false*: Every probability eigenvector of a regular transition matrix has eigenvalue equal to 1.

Solution: ■ ???????? False.

10.4.20. (*a*) Construct an example of a irregular transition matrix. (*b*) Construct an example of a regular transition matrix that has one or more zero entries.

Solution: (a) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; (*b*) $\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{pmatrix}$.

♢ 10.4.21. Let T be a transition matrix. Prove that if $\mathbf{u}$ is a probability vector, so is $\mathbf{v} = T\,\mathbf{u}$.

Solution: The i^{th} entry of $\mathbf{u}^{(k+1)}$ is $u_i^{(k+1)} = \displaystyle\sum_{j=1}^{n} t_{ij}\, u_j^{(k)}$. Since each $t_{ij} \geq 0$ and $u_j^{(k)} \geq 0$, the sum $u_i^{(k+1)} \geq 0$ also. Moreover, $\displaystyle\sum_{i=1}^{n} u_i^{(k+1)} = \sum_{i,j=1}^{n} t_{ij}\, u_j^{(k)} = \sum_{j=1}^{n} u_j^{(k)} = 1$ because all the column sums of T are equal to 1, and $\mathbf{u}^{(k)}$ is a probability vector. *Q.E.D.*

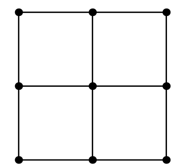
♢ 10.4.22. (*a*) Prove that if T and S are transition matrices, so is their product $T\,S$. (*b*) Prove that if T is a transition matrix, so is T^k for any $k \geq 0$.

Solution:

(a) The columns of TS are obtained by multiplying T by the columns of S. Since S is a transition matrix, its columns are probability vectors. Exercise 10.4.21 shows that each column of TS is also a probability vector, and so the product is a transition matrix.

(b) This follows by induction from part (a), where we write $T^{k+1} = T\,T^k$.

♣ 10.4.23. A bug crawls along the edges of the pictured square lattice with nine vertices. Upon arriving at a vertex, there is an equal probability of it choosing any edge to leave the vertex or stayilng at the vertex. Set up the Markov chain described by the bug's motion, and determine how often, on average, it visits each vertex.

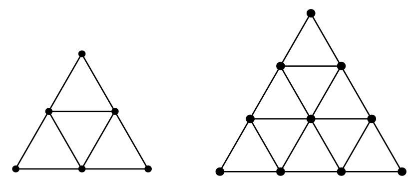


Solution: ■

♣ 10.4.24. (a) Repeat Exercise 10.4.23 for a bug on a square lattice with four vertices to a side. (b) Experiment with a square lattice with n vertices on each side for n large. Do you notice any form of limiting behavior?

Solution: ■

♣ 10.4.25. (a) Repeat Exercise 10.4.23 for the pictured triangular lattices. (b) Investigate what happens as the number of vertices in the triangular lattice gets larger and larger.



Solution: ■

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10.5. Iterative Solution of Linear Algebraic Systems.

10.5.1. (a) Find the spectral radius of the matrix $T = \begin{pmatrix} 1 & 1 \\ -1 & -\frac{7}{6} \end{pmatrix}$. ($b$) Predict the long term behavior of the iterative system $\mathbf{u}^{(k+1)} = T\,\mathbf{u}^{(k)} + \mathbf{b}$ where $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ in as much detail as you can.

Solution:

(a) The eigenvalues are $-\frac{1}{2}, \frac{1}{3}$, so $\rho(T) = \frac{1}{2}$.

(b) The iterates will converge to the fixed point $\left(-\frac{1}{6}, 1\right)^T$ at rate $\frac{1}{2}$. Asymptotically, they come in to the fixed point along the direction of the dominant eigenvector $(-3, 2)^T$.

10.5.2. Answer Exercise 10.5.1 when (a) $T = \begin{pmatrix} 1 & -\frac{1}{2} \\ -1 & \frac{3}{2} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;

(b) $T = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \\ 1 & 1 & \frac{1}{4} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$; ($c$) $T = \begin{pmatrix} -.05 & .15 & .15 \\ .35 & .15 & -.35 \\ -.2 & -.2 & .3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -1.5 \\ 1.6 \\ 1.7 \end{pmatrix}$.

Solution:

(a) $\rho(T) = 2$. The iterates diverge: $\|\,\mathbf{u}^{(k)}\,\| \to \infty$ at a rate of 2.

(b) $\rho(T) = \frac{3}{4}$. The iterates converge to the fixed point $(1.6, 0.8, 7.2)^T$ at a rate $\frac{3}{4}$, along the dominant eigenvector direction $(1,2,6)^T$.

(*c*) $\rho(T) = \frac{1}{2}$. The iterates converge to the fixed point $(-1, .4, 2.6)^T$ at a rate $\frac{1}{2}$, along the dominant eigenvector direction $(0,-1,1)^T$.

10.5.3. Which of the following systems have a diagonally dominant coefficient matrix?

(a) $\begin{aligned} 5x - y &= 1, \\ -x + 3y &= -1; \end{aligned}$ (*b*) $\begin{aligned} \tfrac{1}{2}x + \tfrac{1}{3}y &= 1, \\ \tfrac{1}{5}x + \tfrac{1}{4}y &= 6; \end{aligned}$ (*c*) $\begin{aligned} -5x + y &= 3, \\ -3x + 2y &= -2; \end{aligned}$ (*d*) $\begin{aligned} -2x + y + z &= 1, \\ -x + 2y - z &= -2, \\ x - y + 3z &= 1; \end{aligned}$

(*e*) $\begin{aligned} -x + \tfrac{1}{2}y + \tfrac{1}{3}z &= 1, \\ \tfrac{1}{3}x + 2y + \tfrac{3}{4}z &= -3, \\ \tfrac{2}{3}x + \tfrac{1}{4}y - \tfrac{3}{2}z &= 2; \end{aligned}$ (*f*) $\begin{aligned} x - 2y + z &= 1, \\ -x + 2y + z &= -1, \\ x + 3y - 2z &= 3; \end{aligned}$ (*g*) $\begin{aligned} -4x + 2y + z &= 2, \\ -x + 3y + z &= -1, \\ x + 4y - 6z &= 3. \end{aligned}$

Solution: (a,b,e,g) are diagonally dominant.

♠ 10.5.4. For the diagonally dominant systems in Exercise 10.5.3, starting with the initial guess $x = y = z = 0$, compute the solution to 2 decimal places using the Jacobi method. Check your answer by solving the system directly by Gaussian Elimination.

Solution: (*a*) $x = \frac{1}{7} = .142857,\ \ y = -\frac{2}{7} = -.285714;$ (*b*) $x = -30,\ \ y = 48;$
(*e*) $x = -1.9172,\ \ y = -.339703,\ \ z = -2.24204;$
(*g*) $x = -.84507,\ \ y = -.464789,\ \ z = -.450704;$

♠ 10.5.5. (*a*) Do any of the non-diagonally dominant systems in Exercise 10.5.3 lead to convergent Jacobi schemes? *Hint*: Check the spectral radius of the Jacobi matrix. (*b*) For the convergent systems in Exercise 10.5.3, starting with the initial guess $x = y = z = 0$, compute the solution to 2 decimal places using the Jacobi method, and check your answer by solving the system directly by Gaussian Elimination.

Solution: (*c*) Jacobi spectral radius = .547723, so Jacobi converges to the solution
$x = \frac{8}{7} = 1.142857,\ \ y = \frac{19}{7} = 2.71429;$
(*d*) Jacobi spectral radius = .5, so Jacobi converges to the solution
$x = -\frac{10}{9} = -1.1111,\ \ y = -\frac{13}{9} = -1.4444,\ \ z = \frac{2}{9} = .2222;$
(*f*) Jacobi spectral radius = 1.1180, so Jacobi does not converge.

10.5.6. The following linear systems have positive definite coefficient matrices. Use the Jacobi method starting with $\mathbf{u}^{(0)} = \mathbf{0}$ to find the solution to 4 decimal place accuracy.

(a) $\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, (*c*) $\begin{pmatrix} 6 & -1 & -3 \\ -1 & 7 & 4 \\ -3 & 4 & 9 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}$,

(*d*) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 5 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ -5 \\ 0 \end{pmatrix}$, (*e*) $\begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, (*f*) $\begin{pmatrix} 3 & 1 & 0 & -1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ -1 & 0 & 1 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}$.

Solution: (a) $\mathbf{u} = \begin{pmatrix} .7857 \\ .3571 \end{pmatrix}$, (*b*) $\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, (*c*) $\mathbf{u} = \begin{pmatrix} .3333 \\ -1.0000 \\ 1.3333 \end{pmatrix}$,

(*d*) $\mathbf{u} = \begin{pmatrix} .7273 \\ -3.1818 \\ .6364 \end{pmatrix}$, (*e*) $\mathbf{u} = \begin{pmatrix} .8750 \\ -.1250 \\ -.1250 \\ -.1250 \end{pmatrix}$, (*f*) $\mathbf{u} = \begin{pmatrix} 0. \\ .7143 \\ -.1429 \\ -.2857 \end{pmatrix}$.

♣ 10.5.7. Let A be the $n \times n$ tridiagonal matrix with all its diagonal entries equal to c and all 1's on the sub- and super-diagonals. (*a*) For which values of c is A diagonally dominant?

(b) For which values of c does the Jacobi iteration for $A\mathbf{x} = \mathbf{b}$ converge to the solution? What is the rate of convergence? *Hint*: Use Exercise 8.2.47. (c) Set $c = 2$ and use the Jacobi method to solve the linear systems $K\mathbf{u} = \mathbf{e}_1$, for $n = 5, 10$, and 20. Starting with an initial guess of $\mathbf{0}$, how many Jacobi iterations does it take to obtain 3 decimal place accuracy? Does the convergence rate agree with what you computed in part (c)?

Solution: (a) $|\,c\,| > 2$. (b) If $c = 0$, then $D = c\,\mathrm{I} = \mathrm{O}$ and the Jacobi iteration isn't even defined. Otherwise, $T = -D^{-1}(L + U)$ is tridiagonal with diagonal entries all 0 and sub- and super-diagonal entries equal to $-1/c$. According to Exercise 8.2.47, the eigenvalues are $-\dfrac{2}{c}\cos\dfrac{k\pi}{n+1}$ for $k = 1, \ldots, n$, and so the spectral radius is $\rho(T) = \dfrac{2}{|\,c\,|}\cos\dfrac{1}{n+1}$. Thus, convergence requires $|\,c\,| > 2\cos\dfrac{1}{n+1}$; in particular, $|\,c\,| \geq 2$ will ensure convergence for any n.

(c) For $n = 5$ the solution is $\mathbf{u} = (\,.8333, -.6667, .5000, -.3333, .1667\,)^T$ with a convergence rate of $\rho(T) = \cos\frac{1}{6}\pi = .8660$. It takes 51 iterations to obtain 3 decimal place accuracy, while $\log(.5\times 10^{-4})/\log\rho(T) \approx 53$.

For $n = 10$ the solution is $\mathbf{u} = (.9091, -.8182, .7273, -.6364, .5455, -.4545, .3636, -.2727, .1818, -.0909)^T$ with a convergence rate of $\cos\frac{1}{11}\pi = .9595$. It takes 173 iterations to obtain 3 decimal place accuracy, while $\log(.5\times 10^{-4})/\log\rho(T) \approx 184$.

For $n = 20$ the solution is $\mathbf{u} = (.9524, -.9048, .8571, -.8095, .7619, -.7143, .6667, -.6190, .5714, -.5238, .4762, -.4286, .3810, -.3333, .2857, -.2381, .1905, -.1429, .0952, -.0476)^T$ with a convergence rate of $\cos\frac{1}{21}\pi = .9888$. It takes 637 iterations to obtain 3 decimal place accuracy, while $\log(.5\times 10^{-4})/\log\rho(T) \approx 677$.

10.5.8. Prove that $\mathbf{0} \neq \mathbf{u} \in \ker A$ if and only if $\mathbf{u}$ is a eigenvector of the Jacobi iteration matrix with eigenvalue 1. What does this imply about convergence?

Solution: If $A\mathbf{u} = \mathbf{0}$, then $D\mathbf{u} = -(L+U)\mathbf{u}$, and hence $T\mathbf{u} = -D^{-1}(L+U)\mathbf{u} = \mathbf{u}$. Therefore $\mathbf{u}$ is a eigenvector for T with eigenvalue 1. Therefore, $\rho(T) \geq 1$, which implies that T is *not* a convergent matrix.

◇ 10.5.9. Prove that if A is a nonsingular coefficient matrix, then one can always arrange that all its diagonal entries are nonzero by suitably permuting its rows.

Solution: If A is nonsingular, then at least one of the terms in the general determinant expansion (1.84) is nonzero. If $a_{1,\pi(1)}\,a_{2,\pi(2)}\,\cdots\,a_{n,\pi(n)} \neq 0$ then each $a_{i,\pi(i)} \neq 0$. Applying the permutation π to the rows of A will produce a matrix whose diagonal entries are all nonzero.

10.5.10. Consider the iterative system (10.53) with spectral radius $\rho(T) < 1$. Explain why it takes roughly $-1/\log_{10}\rho(T)$ iterations to produce one further decimal digit of accuracy in the solution.

Solution: Assume, for simplicity, that T is complete with a single dominant eigenvalue λ_1 so that $\rho(T) = |\,\lambda_1\,|$. We expand the initial error $\mathbf{e}^{(0)} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ in terms of its eigenvectors. Then $\mathbf{e}^{(k)} = T^k\mathbf{e}^{(0)} = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_n\lambda_n^k\mathbf{v}_n$, which, for $k \gg 0$, is approximately $\mathbf{e}^{(k)} \approx c_1\lambda_1^k\mathbf{v}_1$. Thus, $\|\,\mathbf{e}^{(k+j)}\,\| \approx \rho(T)^j\,\|\,\mathbf{e}^{(k)}\,\|$. In particular, if at iteration number k we have m decimal places of accuracy, so $\|\,\mathbf{e}^{(k)}\,\| \leq .5\times 10^{-m}$, then, approximately, $\|\,\mathbf{e}^{(k+j)}\,\| \leq .5\times 10^{-m+j\log_{10}\rho(T)} = .5\times 10^{-m-1}$ provided $j = -1/\log_{10}\rho(T)$. *Q.E.D.*

10.5.11. *True or false*: If a system $A\mathbf{u} = \mathbf{b}$ has diagonally dominant coefficient matrix A, then the equivalent system obtained by applying an elementary row operation to A also has diagonally dominant coefficient matrix.

Solution: False for elementary row operations of types 1 & 2, but true for those of type 3.

♡ 10.5.12. Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 4 & 1 & -2 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \\ 4 \end{pmatrix}$.

(*a*) First, solve the equation directly by Gaussian Elimination. (*b*) Using the initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$, carry out three iterations of the Jacobi algorithm to compute $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}$ and $\mathbf{x}^{(3)}$. How close are you to the exact solution? (*c*) Write the Jacobi iteration in the form $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$. Find the 3×3 matrix T and the vector c explicitly. (*d*) Using the initial approximation $\mathbf{x}^{(0)} = \mathbf{0}$, carry out three iterations of the Gauss–Seidel algorithm. Which is a better approximation to the solution — Jacobi or Gauss–Seidel? (*e*) Write the Gauss–Seidel iteration in the form $\mathbf{x}^{(k+1)} = \widetilde{T}\mathbf{x}^{(k)} + \mathbf{c}$. Find the 3×3 matrix T and the vector c explicitly. (*f*) Determine the spectral radius of the Jacobi matrix T, and use this to prove that the Jacobi method iteration will converge to the solution of $A\mathbf{x} = \mathbf{b}$ for any choice of the initial approximation $\mathbf{x}^{(0)}$. (*g*) Determine the spectral radius of the Gauss–Seidel matrix T. Which method converges faster? (*h*) For the faster method, how many iterations would you expect to need to obtain 5 decimal place accuracy? (*i*) Test your prediction by computing the solution to the desired accuracy.

Solution:

(*a*) $\mathbf{x} = \begin{pmatrix} \frac{7}{23} \\ \frac{6}{23} \\ \frac{40}{23} \end{pmatrix} = \begin{pmatrix} .30435 \\ .26087 \\ 1.73913 \end{pmatrix}$;

(*b*) $\mathbf{x}^{(1)} = \begin{pmatrix} -.5 \\ -.25 \\ 1.75 \end{pmatrix}$, $\mathbf{x}^{(2)} = \begin{pmatrix} .4375 \\ .0625 \\ 1.8125 \end{pmatrix}$, $\mathbf{x}^{(3)} = \begin{pmatrix} .390625 \\ .3125 \\ 1.65625 \end{pmatrix}$, with error $\mathbf{e}^{(3)} = \begin{pmatrix} .0862772 \\ .0516304 \\ -.0828804 \end{pmatrix}$;

(*c*) $\mathbf{x}^{(k+1)} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 \end{pmatrix} \mathbf{x}^{(k)} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ \frac{7}{4} \end{pmatrix}$;

(*d*) $\mathbf{x}^{(1)} = \begin{pmatrix} -.5 \\ -.375 \\ 1.78125 \end{pmatrix}$, $\mathbf{x}^{(2)} = \begin{pmatrix} .484375 \\ .316406 \\ 1.70801 \end{pmatrix}$, $\mathbf{x}^{(3)} = \begin{pmatrix} .274902 \\ .245728 \\ 1.74271 \end{pmatrix}$; the error at the third iteration is $\mathbf{e}^{(3)} = \begin{pmatrix} -.029446 \\ -.015142 \\ .003576 \end{pmatrix}$, which is about 30% of the Jacobi error;

(*e*) $\mathbf{x}^{(k+1)} = \begin{pmatrix} 0 & -\frac{1}{4} & \frac{1}{2} \\ 0 & -\frac{1}{16} & \frac{3}{8} \\ 0 & \frac{3}{64} & -\frac{1}{32} \end{pmatrix} \mathbf{x}^{(k)} + \begin{pmatrix} -\frac{1}{2} \\ -\frac{3}{8} \\ \frac{57}{32} \end{pmatrix}$;

(*f*) $\rho(T_J) = \frac{\sqrt{3}}{4} = .433013$, $\rho(T_{GS}) = \frac{3+\sqrt{73}}{64} = .180375$, so Gauss–Seidel converges about $\log \rho_{GS} / \log \rho_J = 2.046$ times as fast.

(*g*) Approximately $\log(.5 \times 10^{-6}) / \log \rho_{GS} \approx 8.5$ iterations. Indeed, under Gauss–Seidel, $\mathbf{x}^{(9)} = \begin{pmatrix} .304347 \\ .260869 \\ 1.73913 \end{pmatrix}$, with error $\mathbf{e}^{(9)} = 10^{-6} \begin{pmatrix} -1.0475 \\ -.4649 \\ .1456 \end{pmatrix}$.

♠ 10.5.13. For the diagonally dominant systems in Exercise 10.5.3, starting with the initial guess $x = y = z = 0$, compute the solution to 3 decimal places using the Gauss–Seidel method. Check your answer by solving the system directly by Gaussian Elimination.

Solution: (*a*) $x = \frac{1}{7} = .142857$, $y = -\frac{2}{7} = -.285714$; (*b*) $x = -30$, $y = 48$;
(*e*) $x = -1.9172$, $y = -.339703$, $z = -2.24204$;

(g) $x = -.84507$, $y = -.464789$, $z = -.450704$;

10.5.14. Which of the systems in Exercise 10.5.3 lead to convergent Gauss–Seidel schemes? In each case, which converges faster, Jacobi or Gauss–Seidel?

Solution: (a) $\rho_J = .2582$, $\rho_{GS} = .0667$; (b) $\rho_J = .7303$, $\rho_{GS} = .5333$; (c) $\rho_J = .5477$, $\rho_{GS} = .3$; (d) $\rho_J = .5$, $\rho_{GS} = .2887$; (e) $\rho_J = .4541$, $\rho_{GS} = .2887$; (f) $\rho_J = .3108$, $\rho_{GS} = .1667$; (g) $\rho_J = 1.118$, $\rho_{GS} = .7071$; Thus, all systems lead to convergent Gauss–Seidel schemes, with faster convergence than Jacobi (which doesn't even converge in case (g)).

10.5.15. (a) Solve the positive definite linear systems in Exercise 10.5.6 using the Gauss–Seidel scheme to achieve 4 decimal place accuracy. (b) Compare the convergence rate with the Jacobi method.

Solution:

(a) Solution: $\mathbf{u} = \begin{pmatrix} .7857 \\ .3571 \end{pmatrix}$; spectral radii: $\rho_J = \frac{1}{\sqrt{15}} = .2582$, $\rho_{GS} = \frac{1}{15} = .06667$, so Gauss–Seidel converges exactly twice as fast;

(b) Solution: $\mathbf{u} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$; spectral radii: $\rho_J = \frac{1}{\sqrt{2}} = .7071$, $\rho_{GS} = \frac{1}{2} = .5$, so Gauss–Seidel converges exactly twice as fast;

(c) Solution: $\mathbf{u} = \begin{pmatrix} .3333 \\ -1.0000 \\ 1.3333 \end{pmatrix}$; spectral radii: $\rho_J = .7291$, $\rho_{GS} = .3104$, so Gauss–Seidel converges $\log \rho_{GS} / \log \rho_J = 3.7019$ times as fast;

(d) Solution: $\mathbf{u} = \begin{pmatrix} .7273 \\ -3.1818 \\ .6364 \end{pmatrix}$; spectral radii: $\rho_J = \frac{2}{\sqrt{15}} = .5164$, $\rho_{GS} = \frac{4}{15} = .2667$, so Gauss–Seidel converges exactly twice as fast;

(e) Solution: $\mathbf{u} = \begin{pmatrix} .8750 \\ -.1250 \\ -.1250 \\ -.1250 \end{pmatrix}$; spectral radii: $\rho_J = .6$, $\rho_{GS} = .1416$, so Gauss–Seidel converges $\log \rho_{GS} / \log \rho_J = 3.8272$ times as fast;

(f) Solution: $\mathbf{u} = \begin{pmatrix} 0. \\ .7143 \\ -.1429 \\ -.2857 \end{pmatrix}$; spectral radii: $\rho_J = .4714$, $\rho_{GS} = .3105$, so Gauss–Seidel converges $\log \rho_{GS} / \log \rho_J = 1.5552$ times as fast.

♣ 10.5.16. Let $A = \begin{pmatrix} c & 1 & 0 & 0 \\ 1 & c & 1 & 0 \\ 0 & 1 & c & 1 \\ 0 & 0 & 1 & c \end{pmatrix}$. ($a$) For what values of c is A diagonally dominant? (b) Use a computer to find the smallest positive value of $c > 0$ for which Jacobi iteration converges. (c) Find the smallest positive value of $c > 0$ for which Gauss–Seidel iteration converges. Is your answer the same? (d) When they both converge, which converges faster — Jacobi or Gauss–Seidel? Does your answer depend upon the value of c?

Solution:

(a) $|\,c\,| > 2$;

(b)

♠ 10.5.17. Consider the linear system

$$2.4\,x - .8\,y + .8\,z = 1, \qquad -.6\,x + 3.6\,y - .6\,z = 0, \qquad 15\,x + 14.4\,y - 3.6\,z = 0.$$

Show, by direct computation, that Jacobi iteration converges to the solution, but Gauss–

Seidel does not.

Solution: The solution is $x = .083799$, $y = .21648$, $z = 1.21508$. The Jacobi spectral radius is .8166, and so it converges reasonably rapidly to the solution; after 50 iterations, $x^{(50)} = .0838107$, $y^{(50)} = .216476$, $z^{(50)} = 1.21514$. On the other hand, the Gauss–Seidel spectral radius is 1.0994, and it slowly diverges; after 50 iterations, $x^{(50)} = -30.5295$, $y^{(50)} = 9.07764$, $z^{(50)} = -90.8959$.

♠ 10.5.18. Discuss convergence of Gauss–Seidel iteration for the system

$$5x + 7y + 6z + 5w = 23, \qquad 6x + 8y + 10z + 9w = 33,$$
$$7x + 10y + 8z + 7w = 32, \qquad 5x + 7y + 9z + 10w = 31.$$

Solution: The solution is $x = y = z = w = 1$. Gauss–Seidel converges, but extremely slowly. After 2000 iterations, the approximate solution $x^{(50)} = 1.00281$, $y^{(50)} = .99831$, $z^{(50)} = .999286$, $w^{(50)} = 1.00042$, correct to 2 decimal places. The spectral radius is .9969 and so it takes, on average, 741 iterations per decimal place.

10.5.19. Let $A = \begin{pmatrix} 2 & 4 & -4 \\ 3 & 3 & 3 \\ 2 & 2 & 1 \end{pmatrix}$. Find the spectral radius of the Jacobi and Gauss–Seidel iteration matrices, and discuss their convergence.

Solution: $\rho(T_J) = 0$ while $\rho(T_{GS}) = 2$. Thus Jacobi converges very rapidly, whereas Gauss–Seidel diverges.

♠ 10.5.20. Consider the linear system $H_5 \mathbf{u} = \mathbf{e}_1$, where H_5 is the 5×5 Hilbert matrix. Does the Jacobi method converge to the solution? If so, how fast? What about Gauss–Seidel?

Solution: Jacobi doesn't converge because its spectral radius is 3.4441. Gauss–Seidel converges, but extremely slowly, since its spectral radius is .999958.

♢ 10.5.21. How many arithmetic operations are needed to perform k steps of the Jacobi iteration? How does this compare with Gaussian Elimination? Do the same conclusions apply to Gauss–Seidel? Under what conditions is Jacobi or Gauss–Seidel more efficient than Gaussian Elimination?

Solution: ■

♣ 10.5.22. The *naïve iterative method* for solving $A\mathbf{u} = \mathbf{b}$ is to rewrite it in fixed point form $\mathbf{u} = T\mathbf{u} + \mathbf{c}$, where $T = \mathrm{I} - A$ and $\mathbf{c} = \mathbf{b}$. (*a*) What conditions on the eigenvalues of A ensure convergence of the naïve method? (*b*) Use the Gerschgorin Theorem 10.34 to prove that the naïve method converges to the solution to $\begin{pmatrix} .8 & -.1 & -.1 \\ .2 & 1.5 & -.1 \\ .2 & -.1 & 1.0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.
(*c*) Check by implementing the method.

Solution:

(*a*) If λ is an eigenvalue of $T = \mathrm{I} - A$, then $\mu = 1 - \lambda$ is an eigenvalue of A, and hence we require the eigenvalues of A to satisfy $|\,1 - \mu\,| < 1$.

(*b*) The Gerschgorin disks are

$$D_1 = \{\, |\,z - .8\,| \le .2 \,\}, \qquad D_2 = \{\, |\,z - 1.5\,| \le .3 \,\}, \qquad D_3 = \{\, |\,z - 1\,| \le .3 \,\},$$

and hence all eigenvalues of A lie within a distance 1 of 1. Indeed, we can explicitly compute the eigenvalues of A, which are

$$\mu_1 = 1.5026, \qquad \mu_2 = .8987 + .1469\,\mathrm{i}, \qquad \mu_3 = .8987 - .1469\,\mathrm{i}\,.$$

Hence, the spectral radius of $T = \mathrm{I} - A$ is $\rho(T) = \max\big\{\, |\,1 - \mu_j\,| \,\big\} = .5026$. Starting the iterations with $\mathbf{u}^{(0)} = \mathbf{0}$, we arrive at the solution $\mathbf{u}^\star = (1.36437, -.73836, 1.65329)^T$ to 4 decimal places after 13 iterations.

♡ 10.5.23. Consider the linear system $A\mathbf{u} = \mathbf{b}$ where $A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.
(*a*) Discuss the convergence of the Jacobi iteration method. (*b*) Discuss the convergence of the Gauss–Seidel iteration method. (*c*) Write down the explicit formulas for the SOR method. (*d*) What is the optimal value of the relaxation parameter ω for this system? How much faster is the convergence as compared to the ordinary Jacobi method? (*e*) Suppose your initial guess is $\mathbf{u}^{(0)} = \mathbf{0}$. Give an estimate as to how many steps each iterative method (Jacobi, Gauss–Seidel, SOR) would require in order to approximate the solution to the system to within 5 decimal places. (*f*) Verify your answer by direct computation.

♣ 10.5.24. Consider the linear system

$$4x - y - z = 1, \quad -x + 4y - w = 2, \quad -x + 4z - w = 0, \quad -y - z + 4w = 1.$$

(*a*) Find the solution using Gaussian Elimination and Back Substitution. (*b*) Using $\mathbf{0}$ as your initial guess, how many iterations are required to approximate the solution to within six decimal places using (*i*) Jacobi iteration, (*ii*) Gauss–Seidel iteration. Can you estimate the spectral radii of the relevant matrices in each case? (*c*) Try to find the solution using the SOR method with parameter ω taking various values between .5 and 1.5. Which value of ω gives the fastest convergence? What is the spectral radius of the SOR matrix?

♠ 10.5.25. (*a*) Find the spectral radius of the Jacobi and Gauss–Seidel iteration matrices when $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$. (*b*) Is A diagonally dominant? (*c*) Use (10.87) to fix the optimal value of the SOR parameter. Verify that the spectral radius of the resulting iteration matrix is given by formula (10.87). (*d*) For each iterative scheme, predict how many iterations are needed to solve the system to 4 decimal places, and then verify your predictions by direct computation.

♠ 10.5.26. Change the matrix in Exercise 10.5.25 to $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$, and answer the same questions. Does the SOR method with parameter given by (10.87) speed the iterations up? Why not? Can you find a value of the SOR parameter that does?

♠ 10.5.27. Let A be the $n \times n$ tridiagonal matrix with all 2's on the main diagonal and all -1's on the sub- and super-diagonal. (*a*) Use Exercise 8.2.46 to find the spectral radius of the Jacobi iteration method to solve $A\mathbf{u} = \mathbf{b}$. (*b*) What is the optimal value of the SOR parameter based on (10.87)? How many Jacobi iterations are needed to match the effect of a single SOR step? (*c*) Test out your conclusions by solving the 8×8 system $A\mathbf{u} = \mathbf{e}_1$ using both Jacobi and SOR to approximate the solution to 3 decimal places.

♠ 10.5.28. In Exercise 10.5.18 you were asked to solve a system by Gauss–Seidel. How much faster can you design an SOR scheme to converge? Experiment with several values of the relaxation parameter ω, and discuss what you find.

♠ 10.5.29. Investigate the three basic iterative techniques — Jacobi, Gauss–Seidel, SOR — for solving the linear system $K^\star \mathbf{u}^\star = \mathbf{f}^\star$ for the cubical circuit in Example 6.4.

♣ 10.5.30. The matrix $A = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$ arises in the finite difference (and finite element) discretization of the Poisson equation on a nine point square grid. Solve the linear system $A\mathbf{u} = \mathbf{e}_5$ using (*a*) Gaussian Elimination; (*b*) Jacobi iteration; (*c*) Gauss–Seidel iteration; (*d*) SOR based on the Jacobi spectral radius.

Solution:

(*a*) $\mathbf{u} = (\,.0625, .125, .0625, .125, .375, .125, .0625, .125, .0625\,)^T$;

(*b*) it takes 11 Jacobi iterations to compute the first two decimal places of the solution, and 17 for 3 place accuracy;

(*c*) it takes 6 Gauss–Seidel iterations to compute the first two decimal places of the solution, and 9 for 3 place accuracy;

(*d*) $\rho_J = \frac{1}{\sqrt{2}}$ for the

♣ 10.5.31. The generalization of Exercise 10.5.30 to an $n \times n$ grid results in an $n^2 \times n^2$ matrix in block tridiagonal form $A = \begin{pmatrix} K & -\mathrm{I} & & & \\ -\mathrm{I} & K & -\mathrm{I} & & \\ & -\mathrm{I} & K & -\mathrm{I} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$, in which K is the tridiagonal $n \times n$ matrix with 4's on the main diagonal and -1's on the sub- and super-diagonal, while I denotes an $n \times n$ identity matrix. Use the known value of the Jacobi spectral radius $\rho_J = \cos \dfrac{\pi}{n+1}$, [**74**], to design an SOR method to solve the linear system $A\mathbf{u} = \mathbf{f}$. Run your method on the cases $n = 5$ and $\mathbf{f} = \mathbf{e}_{13}$ and $n = 25$ and $\mathbf{f} = \mathbf{e}_{63}$ corresponding to a unit force at the center of the grid. How much faster is the convergence rate?

♣ 10.5.32. Consider the linear system $A\mathbf{x} = \mathbf{b}$ based on the $n \times n$ pentadiagonal matrix

$$A = \begin{pmatrix} z & 1 & 1 & 0 & & & \\ 1 & z & 1 & 1 & 0 & & \\ 1 & 1 & z & 1 & 1 & 0 & \ddots \\ 0 & 1 & 1 & z & 1 & 1 & \ddots \\ & 0 & 1 & 1 & z & 1 & \ddots \\ & & 0 & 1 & 1 & z & \ddots \\ & & & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

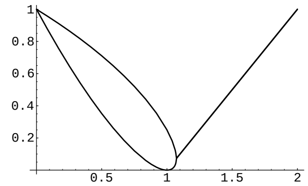
(*a*) For what values of z are the Jacobi and Gauss–Seidel methods guaranteed to converge? (*b*) Test your answer numerically for $n = 20$. (*c*) How small can $|\,z\,|$ be before the methods diverge? (*d*) For $z = 4$, can you predict the spectral radii of the coefficient matrices for both iterative schemes from your numerical data? (*e*) For $z = 4$, find a value of ω for which the Gauss–Seidel SOR method converges significantly faster. (*f*) If possible, determine an optimal value for ω.

◇ 10.5.33. For the matrix treated in Example 10.50, prove that (*a*) as ω increases from 1 to $8 - 4\sqrt{3}$, the two eigenvalues move towards each other, with the larger one decreasing in magnitude; (*b*) if $\omega > 8 - 4\sqrt{3}$, the eigenvalues are complex conjugates, with larger modulus than the optimal value. Can you conclude that $\omega_\star = 8 - 4\sqrt{3}$ is the optimal value for the SOR parameter?

Solution: The two eigenvalues

$$\lambda_1 = \tfrac{1}{8}\left(\omega^2 - 8\omega + 8 + \omega\sqrt{\omega^2 - 16\omega + 16}\,\right), \qquad \lambda_2 = \tfrac{1}{8}\left(\omega^2 - 8\omega + 8 - \omega\sqrt{\omega^2 - 16\omega + 16}\,\right).$$

are real for $0 \le \omega \le 8 - 4\sqrt{3}$. A graph of the modulus of the eigenvalues over the range $0 \le \omega \le 2$



reveals that, as ω increases, the smaller eigenvalue is increasing and the larger decreasing until they meet at $8 - 4\sqrt{3}$; after this point, both eigenvalues are complex conjugates of the same modulus. To prove this analytically, we compute

$$\frac{d\lambda_2}{d\omega} = \frac{3-\omega}{4} + \frac{2-\omega}{\sqrt{\omega^2 - 16\omega + 16}} > 0$$

for $1 \le \omega \le 8 - 4\sqrt{3}$, and so the smaller eigenvalue is increasing. Furthermore,

$$\frac{d\lambda_2}{d\omega} = \frac{3-\omega}{4} - \frac{2-\omega}{\sqrt{\omega^2 - 16\omega + 16}} < 0$$

on the same interval, so the larger eigenvalue is decreasing. Once $\omega > 8 - 4\sqrt{3}$, the eigenvalues are complex conjugates, of modulus $|\,\lambda_1\,| = |\,\lambda_2\,| = \omega - 1 > \omega_\star - 1$. *Q.E.D.*

♡ 10.5.34. If $\mathbf{u}^{(k)}$ is an approximation to the solution to $A\mathbf{u} = \mathbf{b}$, then the *residual vector* $\mathbf{r}^{(k)} = \mathbf{b} - A\mathbf{u}^{(k)}$ measures how accurately the approximation solves the system.

(*a*) Show that the Jacobi iteration can be written in the form $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + D^{-1}\mathbf{r}^{(k)}$.

(*b*) Show that the Gauss–Seidel iteration has the form $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (L + D)^{-1}\mathbf{r}^{(k)}$.

(*c*) Show that the SOR iteration has the form $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (\omega L + D)^{-1}\mathbf{r}^{(k)}$.

(*d*) If $\|\,\mathbf{r}^{(k)}\,\|$ is small, does this mean that $\mathbf{u}^{(k)}$ is close to the solution? Explain your answer and illustrate with a couple of examples.

Solution:

(*a*) $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + D^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - D^{-1}A\mathbf{u}^{(k)} + D^{-1}\mathbf{b} = \mathbf{u}^{(k)} - D^{-1}(L + D + U)u^{(k)} + D^{-1}\mathbf{b} = -\,D^{-1}(L + U)u^{(k)} + D^{-1}\mathbf{b}$, which agrees with (10.66).

(*b*) $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (L + D)^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - (L + D)^{-1}A\mathbf{u}^{(k)} + (L + D)^{-1}\mathbf{b} = \mathbf{u}^{(k)} - (L + D)^{-1}(L + D + U)u^{(k)} + (L + D)^{-1}\mathbf{b} = -\,(L + D)^{-1}U\,u^{(k)} + (L + D)^{-1}\mathbf{b}$, which agrees with (10.72).

(*c*) $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + (\omega L + D)^{-1}\mathbf{r}^{(k)} = \mathbf{u}^{(k)} - (\omega L + D)^{-1}A\mathbf{u}^{(k)} + (\omega L + D)^{-1}\mathbf{b} = \mathbf{u}^{(k)} - (\omega L + D)^{-1}(L + D + U)u^{(k)} + (\omega L + D)^{-1}\mathbf{b} = -\,(\omega L + D)^{-1}\big(\,(1-\omega)D + U\,\big)u^{(k)} + (\omega L + D)^{-1}\mathbf{b}$, which agrees with (10.81).

(*d*) If $\mathbf{u}^\star$ is the exact solution, so $A\mathbf{u}^\star = \mathbf{b}$, then $\mathbf{r}^{(k)} = A(\mathbf{u}^\star - \mathbf{u}^{(k)})$ and so $\|\,\mathbf{u}^{(k)} - \mathbf{u}^\star\,\| \le \|\,A^{-1}\,\|\,\|\,\mathbf{r}^{(k)}\,\|$. Thus, if $\|\,\mathbf{r}^{(k)}\,\|$ is small, the iterate $\mathbf{u}^{(k)}$ is close to the solution $\mathbf{u}^\star$ provided $\|\,A^{-1}\,\|$ is not too large. For instance, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-5} \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{r}^{(k)} =$.

$\blacksquare$

10.5.35. Let K be a positive definite $n \times n$ matrix with eigenvalues $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n > 0$. For what values of ε does the iterative system $\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \varepsilon\,\mathbf{r}^{(k)}$, where $\mathbf{r}^{(k)} = \mathbf{f} - K\mathbf{u}^{(k)}$ is the current *residual vector*, converge to the solution? What is the optimal value of ε, and what is the convergence rate?

Solution: Note that the iteration matrix is $T = \mathrm{I} - \varepsilon A$, which has eigenvalues $1 - \varepsilon\lambda_j$. For

$0 < \varepsilon < \dfrac{2}{\lambda_1}$ the iterations converge. The optimal value is $\varepsilon = \dfrac{2}{\lambda_1 + \lambda_n}$, with spectral radius $\rho(T) = \dfrac{\lambda_1 - \lambda_n}{\lambda_n + \lambda_1}$.

10.5.36. Solve the following linear systems by the conjugate gradient method, keeping track of the residual vectors and solution approximations as you iterate.

$$(a)\ \begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 6 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad (c)\ \begin{pmatrix} 6 & -1 & -3 \\ -1 & 7 & 4 \\ -3 & 4 & 9 \end{pmatrix} \mathbf{u} = \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix},$$

$$(d)\ \begin{pmatrix} 6 & -1 & -1 & 5 \\ -1 & 7 & 1 & -1 \\ -1 & 1 & 3 & -3 \\ 5 & -1 & -3 & 6 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \quad (e)\ \begin{pmatrix} 5 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 5 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solution: In each solution, the last $\mathbf{u}_k$ is the actual solution, with residual $\mathbf{r}_k = \mathbf{f} - K\,\mathbf{u}_k = \mathbf{0}$.

$$(a)\ \mathbf{r}_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} .76923 \\ .38462 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} .07692 \\ -.15385 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} .78571 \\ .35714 \end{pmatrix},$$

$$(b)\ \mathbf{r}_0 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} .5 \\ 0 \\ -1 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} -1 \\ -2 \\ -.5 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} .51814 \\ -.72539 \\ -1.94301 \end{pmatrix},$$

$$\mathbf{r}_2 = \begin{pmatrix} 1.28497 \\ -.80311 \\ .64249 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 1. \\ -1.4 \\ -2.2 \end{pmatrix}.$$

$$(c)\ \mathbf{r}_0 = \begin{pmatrix} -1 \\ -2 \\ 7 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} 2.36658 \\ -4.01995 \\ -.81047 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} -.13466 \\ -.26933 \\ .94264 \end{pmatrix},$$

$$\mathbf{r}_2 = \begin{pmatrix} .72321 \\ .38287 \\ .21271 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} .33333 \\ -1.00000 \\ 1.33333 \end{pmatrix}.$$

$$(d)\ \mathbf{r}_0 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} .2 \\ .4 \\ 0 \\ -.2 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} 1.2 \\ -.8 \\ -.8 \\ -.4 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} .90654 \\ .46729 \\ -.33645 \\ -.57009 \end{pmatrix},$$

$$\mathbf{r}_2 = \begin{pmatrix} -1.45794 \\ -.59813 \\ -.26168 \\ -2.65421 \end{pmatrix}, \ \mathbf{u}_3 = \begin{pmatrix} 4.56612 \\ .40985 \\ -2.92409 \\ -5.50820 \end{pmatrix}, \qquad \mathbf{r}_3 = \begin{pmatrix} -1.36993 \\ 1.11307 \\ -3.59606 \\ .85621 \end{pmatrix}, \ \mathbf{u}_4 = \begin{pmatrix} 9.50 \\ 1.25 \\ -10.25 \\ -13.00 \end{pmatrix}.$$

$$(e)\ \mathbf{r}_0 = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{u}_1 = \begin{pmatrix} .8 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_1 = \begin{pmatrix} 0. \\ -.8 \\ -.8 \\ -.8 \end{pmatrix}, \ \mathbf{u}_2 = \begin{pmatrix} .875 \\ -.125 \\ -.125 \\ -.125 \end{pmatrix}.$$

10.5.37. The exact solution to the linear system

$$x + \tfrac{1}{2}\,y + \tfrac{1}{3}\,z = \tfrac{1}{3}, \qquad \tfrac{1}{2}\,x + \tfrac{1}{3}\,y + \tfrac{1}{4}\,z = \tfrac{1}{12}, \qquad \tfrac{1}{3}\,x + \tfrac{1}{4}\,y + \tfrac{1}{5}\,z = \tfrac{1}{30},$$

is $x = 1, y = -2, z = 1$. (*a*) Apply Gaussian Elimination with 2 digit rounding arithmetic to compute the solution. What is your error? (*b*) Does pivoting help? (*c*) Apply the conjugate gradient method using the same 2 digit rounding arithmetic. How much closer is your final solution?

Solution: ■

♣ 10.5.38. According to Example 3.33, the $n \times n$ Hilbert matrix H_n is positive definite, and hence one can apply the conjugate gradient method to solve the linear system $H_n \mathbf{u} = \mathbf{f}$. For the values $n = 5, 10, 30$, let $\mathbf{u}^\star \in \mathbb{R}^n$ be the vector with all entries equal to 1. (*a*) Compute $\mathbf{f} = H_n \mathbf{u}^\star$. (*b*) Use Gaussian Elimination to solve $H_n \mathbf{u} = \mathbf{f}$. How close is your solution to $\mathbf{u}^\star$? (*c*) Does pivoting improve the solution in part (*b*)? (*d*) Use conjugate gradients to solve the same system. How many iterates do you need to obtain a reasonable approximation, say to 2 decimal places, to the exact solution $\mathbf{u}^\star$?

Solution: ■

10.5.39. Try applying the Conjugate Gradient Algorithm to the system
$-x + 2y + z = -2,\ \ y + 2z = 1,\ \ 3x + y - z = 1.$ Do you obtain the solution? Why?

Solution: $\mathbf{r}_0 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} .9231 \\ -.4615 \\ -.4615 \end{pmatrix}$, $\mathbf{r}_1 = \begin{pmatrix} .3077 \\ 2.3846 \\ -1.7692 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 2.7377 \\ -3.0988 \\ -.2680 \end{pmatrix}$,
$\mathbf{r}_2 = \begin{pmatrix} 7.2033 \\ 4.6348 \\ -4.3823 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 5.5113 \\ -9.1775 \\ .7262 \end{pmatrix}$, but the solution is $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. the problem is that the coefficient matrix is not positive definite, and so the fact that the solution is "orthogonal" to the conjugate vectors does not uniquely specify it.

♣ 10.5.40. Use the conjugate gradient method to solve the system in Exercise 10.5.30. How many iterations do you need to obtain the solution that is accurate to 2 decimal places? How does this compare to the Jacobi and SOR methods?

Solution: Remarkably, after only two iterations, the method finds the exact solution: $\mathbf{u}_3 = \mathbf{u}^\star = (\,.0625, .125, .0625, .125, .375, .125, .0625, .125, .0625\,)^T$, and hence the convergence is dramatically faster than the other iterative methods.

♢ 10.5.41. In (10.91), find the value of t_k that minimizes $p(\mathbf{u}_{k+1})$.

♠ 10.5.42. Use the direct gradient descent algorithm (10.91) using the value of t_k found in Exercise 10.5.41 to solve the linear systems in Exercise 10.5.36. Compare the speed of convergence with that of the conjugate gradient method.

10.5.43. *True or false*: If the residual vector satisfies $\|\mathbf{r}\| < .01$, then $\mathbf{u}$ approximates the solution to within two decimal places.

Solution: False. For example, consider the homogeneous systme $K\mathbf{u} = \mathbf{0}$ where $K = \begin{pmatrix} .0001 & 0 \\ 0 & 1 \end{pmatrix}$, with solution $\mathbf{u}^\star = \mathbf{0}$. The residual for $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is $\mathbf{r} = -K\mathbf{u} = \begin{pmatrix} -.01 \\ 0 \end{pmatrix}$ with $\|\mathbf{r}\| = .01$, yet not even the leading digit of $\mathbf{u}$ agrees with the true solution. In general, if $\mathbf{u}^\star$ is the true solution to $K\mathbf{u} = \mathbf{f}$, then the residual is $\mathbf{r} = \mathbf{f} - A\mathbf{u} = A(\mathbf{u}^\star - \mathbf{u})$, and hence $\|\mathbf{u}^\star - \mathbf{u}\| \le \|A^{-1}\|\|\mathbf{r}\|$, so the result is true when $\|A^{-1}\| \le 1$.

10.5.44. How many arithmetic operations are needed to implement one iteration of the conjugate gradient method? How many iterations can you perform before the method becomes more work that direct Gaussian Elimination? *Remark*: If the matrix is sparse, the number of operations can decrease dramatically.

numev **10.6. Numerical Computation of Eigenvalues.**

♠ 10.6.1. Use the power method to find the dominant eigenvalue and associated eigenvector of the following matrices:

$(a)\ \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix},\quad (b)\ \begin{pmatrix} -5 & 2 \\ -3 & 0 \end{pmatrix},\quad (c)\ \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix},\quad (d)\ \begin{pmatrix} -2 & 0 & 1 \\ -3 & -2 & 0 \\ -2 & 5 & 4 \end{pmatrix},$

$(e)\ \begin{pmatrix} -1 & -2 & -2 \\ 1 & 2 & 5 \\ -1 & 4 & 0 \end{pmatrix},\ (f)\ \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 2 & 2 & 2 \end{pmatrix},\ (g)\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix},\ (h)\ \begin{pmatrix} 4 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \end{pmatrix}.$

Solution: In all cases, we use the normalized version (10.102) starting with $\mathbf{u}^{(0)} = \mathbf{e}_1$; the answers are correct to 4 decimal places.

(a) After 17 iterations, we find $\lambda = 2.00002$, $\mathbf{u} = (\,-.55470, .83205\,)^T$;
(b) after 26 iterations, we find $\lambda = -3.00003$, $\mathbf{u} = (\,.70711, .70710\,)^T$;
(c) after 38 iterations, we find $\lambda = 3.99996$, $\mathbf{u} = (\,.57737, -.57735, .57734\,)^T$;
(d) after 121 iterations, we find $\lambda = -3.30282$, $\mathbf{u} = (\,.35356, .81416, -.46059\,)^T$;
(e) after 36 iterations, we find $\lambda = 5.54911$, $\mathbf{u} = (\,-.39488, .71005, .58300\,)^T$;
(f) after 9 iterations, we find $\lambda = 5.23607$, $\mathbf{u} = (\,.53241, .53241, .65810\,)^T$;
(g) after 36 iterations, we find $\lambda = 3.61800$, $\mathbf{u} = (\,.37176, -.60151, .60150, -.37174\,)^T$;
(h) after 30 iterations, we find $\lambda = 5.99997$, $\mathbf{u} = (\,.50001, .50000, .50000, .50000\,)^T$.

♠ 10.6.2. Let T_n be the tridiagonal matrix whose diagonal entries are all equal to 2 and whose sub- and super-diagonal entries all equal 1. Use the power method to find the dominant eigenvalue of T_n for $n = 10, 20, 50$. Do your values agree with those in Exercise 8.2.46. How many iterations do you require to obtain 4 decimal place accuracy?

Solution:
For $n = 10$ it takes 159 iteratations to get $\lambda_1 = 3.9189 = 2 + 2\cos\frac{1}{6}\pi$ to 4 decimal places.
for $n = 20$ it takes 510 iteratations to get $\lambda_1 = 3.9776 = 2 + 2\cos\frac{1}{21}\pi$ to 4 decimal places.
for $n = 50$ it takes 2392 iteratations to get $\lambda_1 = 3.9962 = 2 + 2\cos\frac{1}{51}\pi$ to 4 decimal places.

♠ 10.6.3. Use the power method to find the largest singular value of the following matrices:

$(a)\ \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix},\quad (b)\ \begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 1 \end{pmatrix},\quad (c)\ \begin{pmatrix} 2 & 2 & 1 & -1 \\ 1 & -2 & 0 & 1 \end{pmatrix},\quad (d)\ \begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 1 \end{pmatrix}.$

Solution: In each case, to find the dominant singular value of a matrix A, we apply the power method to $K = A^T A$ and take the square root of the dominant eigenvalue to find the dominant singular value $\sigma_1 = \sqrt{\lambda_1}$.

(a) $K = \begin{pmatrix} 2 & -1 \\ -1 & 13 \end{pmatrix}$; after 11 iterations we obtain $\lambda_1 = 13.0902$ and $\sigma_1 = 3.6180$;

(b) $K = \begin{pmatrix} 8 & -4 & -4 \\ -4 & 10 & 2 \\ -4 & 2 & 2 \end{pmatrix}$; after 15 iterations we obtain $\lambda_1 = 14.4721$ and $\sigma_1 = 3.8042$;

(c) $K = \begin{pmatrix} 5 & 2 & 2 & -1 \\ 2 & 8 & 2 & -4 \\ 2 & 2 & 1 & -1 \\ -1 & -4 & -1 & 2 \end{pmatrix}$; after 16 iterations we obtain $\lambda_1 = 11.6055$ and $\sigma_1 = 3.4067$;

(d) $K = \begin{pmatrix} 14 & -1 & 1 \\ -1 & 6 & -6 \\ 1 & -6 & 6 \end{pmatrix}$; after 39 iterations we obtain $\lambda_1 = 14.7320$ and $\sigma_1 = 3.8382$.

10.6.4. Prove that, for the iterative scheme (10.102), $\| A\mathbf{u}^{(k)} \| \to |\lambda_1|$. Assuming λ_1 is real, explain how to deduce its sign.

Solution: Since $\mathbf{v}^{(k)} \to \lambda_1^k \mathbf{v}_1$ as $k \to \infty$,

$$\mathbf{u}^{(k)} = \frac{\mathbf{v}^{(k)}}{\| \mathbf{v}^{(k)} \|} \to \frac{c_1 \lambda_1^k \mathbf{v}_1}{|c_1|\,|\lambda_1|^k \|\mathbf{v}_1\|} = \begin{cases} \mathbf{u}_1, & \lambda_1 > 0, \\ (-1)^k \mathbf{u}_1, & \lambda_1 < 0, \end{cases} \qquad \text{where} \qquad \mathbf{u}_1 = \operatorname{sign} c_1 \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$$

is one of the two real unit eigenvectors. Thus, $A\mathbf{u}^{(k)} \to \begin{cases} \lambda_1 \mathbf{u}_1, & \lambda_1 > 0, \\ (-1)^k \lambda_1 \mathbf{u}_1, & \lambda_1 < 0, \end{cases}$ so $\| A\mathbf{u}^{(k)} \| \to |\lambda_1|$. If $\lambda_1 > 0$, the iterates $\mathbf{u}^{(k)} \to \mathbf{u}_1$ converge to a unit eigenvector, whereas if $\lambda_1 < 0$, the iterates $\mathbf{u}^{(k)} \to (-1)^k \mathbf{u}_1$ swtich back and forth between the two real unit eigenvectors.

♢ 10.6.5. *The Inverse Power Method.* Let A be a nonsingular matrix. Show that the eigenvalues of A^{-1} are the reciprocals $1/\lambda$ of the eigenvalues of A. How are the eigenvectors related?(a) Show how to use the power method on A^{-1} to produce the smallest (in modulus) eigenvalue of A. (b) What is the rate of convergence of the algorithm? (c) Design a practical iterative algorithm based on the (permuted) LU decomposition of A.

Solution:

(a) If $A\mathbf{v} = \lambda \mathbf{v}$ then $A^{-1}\mathbf{v} = \dfrac{1}{\lambda}\mathbf{v}$, and so $\mathbf{v}$ is also the eigenvector of A^{-1}.

(b) If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, with $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| > 0$ (recalling that 0 cannot be an eigenvalue if A is nonsingular), then $\dfrac{1}{\lambda_1}, \ldots, \dfrac{1}{\lambda_n}$ are the eigenvalues of A^{-1}, and $\dfrac{1}{|\lambda_n|} > \dfrac{1}{|\lambda_{n-1}|} > \cdots > \dfrac{1}{|\lambda_1|}$ and so $\dfrac{1}{\lambda_1}$ is the dominant eigenvalue of A^{-1}. Thus, applying the power method to A^{-1} will produce the reciprocal of the smallest (in absolute value) eigenvalue of A and the corresponding eigenvector.

(c) The rate of convergence of the algorithm is the ratio $|\lambda_n/\lambda_{n-1}|$ of the moduli of the smallest two eigenvalues.

(d) Once we factor $PA = LU$, we can solve the iteration equation $A\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}$ by rewriting it in the form $LU\mathbf{u}^{(k+1)} = P\mathbf{u}^{(k)}$, and then using Forward and Back Substitution to solve for $\mathbf{u}^{(k+1)}$. As you know, this is much faster than computing A^{-1}.

♠ 10.6.6. Apply the inverse power method of Exercise 10.6.7 to the find the smallest eigenvalue of the matrices in Exercise 10.6.1.

Solution:

(a) After 15 iterations, we find $\lambda = .99998$, $\mathbf{u} = (.70711, -.70710)^T$;

(b) after 24 iterations, we find $\lambda = -1.99991$, $\mathbf{u} = (-.55469, -.83206)^T$;

(c) after 12 iterations, we find $\lambda = 1.00001$, $\mathbf{u} = (.40825, .81650, .40825)^T$;

(d) after 6 iterations, we find $\lambda = .30277$, $\mathbf{u} = (.35355, -.46060, .81415)^T$;

(e) after 7 iterations, we find $\lambda = -.88536$, $\mathbf{u} = (-.88751, -.29939, .35027)^T$;

(f) after 7 iterations, we find $\lambda = .76393$, $\mathbf{u} = (.32348, .25561, -.91106)^T$;

(g) after 11 iterations, we find $\lambda = .38197$, $\mathbf{u} = (\,.37175, .60150, .60150, .37175\,)^T$;
(h) after 16 iterations, we find $\lambda = 2.00006$, $\mathbf{u} = (\,.500015, -.50000, .499985, -.50000\,)^T$.

♢ 10.6.7. *The Shifted Inverse Power Method*. Suppose that μ is *not* an eigenvalue of A. Show that the iterative scheme $\mathbf{u}^{(k+1)} = (A - \mu\,\mathrm{I}\,)^{-1}\mathbf{u}^{(k)}$ converges to the eigenvector of A corresponding to the eigenvalue $\lambda^\star$ that is *closest* to μ. Explain how to find the eigenvalue $\lambda^\star$.(a) What is the rate of convergence of the algorithm? (b) What happens if μ is an eigenvalue?

Solution:
(a) According to Exercises 8.2.18, 23, if A has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $(A - \mu\,\mathrm{I}\,)^{-1}$ has eigenvalues $\nu_i = \dfrac{1}{\lambda_i - \mu}$. Thus, applying the power method to $(A - \mu\,\mathrm{I}\,)^{-1}$ will produce its dominant eigenvalue $\nu^\star$, for which $|\,\lambda^s tar - \mu\,|$ is the smallest. We then recover the eigenvalue $\lambda^\star = \mu + \dfrac{1}{\nu^\star}$ of A which is closest to μ.
(b) The rate of convergence is the ratio $|\,(\lambda_i - \mu)/(\lambda_j - \mu)\,|$ of the moduli of the smallest two eigenvalues of the shifted matrix.
(c) μ is an eigenvalue of A if and only if $A - \mu\,\mathrm{I}$ is a singular matrix, and hence one cannot implemment the method. Also choosing μ too close to an eigenvalue will result in an ill-conditioned matrix, and so the algorithm may not converge properly.

♠ 10.6.8. Apply the shifted inverse power method of Exercise 10.6.7 to the find the eigenvalue closest to $\mu = .5$ of the matrices in Exercise 10.6.1.

Solution:
(a) After 11 iterations, we find $\nu^\star = 2.00002$, so $\lambda^\star = 1.0000$, $\mathbf{u} = (\,.70711, -.70710\,)^T$;
(b) after 27 iterations, we find $\nu^\star = -.40003$, so $\lambda = -1.9998$, $\mathbf{u} = (\,.55468, .83207\,)^T$;
(c) after 10 iterations, we find $\nu^\star = 2.00000$, so $\lambda = 1.00000$, $\mathbf{u} = (\,.40825, .81650, .40825\,)^T$;
(d) after 7 iterations, we find $\nu^\star = -5.07037$, so $\lambda = .30278$, $\mathbf{u} = (\,-.35355, .46060, -.81415\,)^T$;
(e) after 8 iterations, we find $\nu^\star = .72183$, so $\lambda = -.88537$, $\mathbf{u} = (\,.88753, .29937, -.35024\,)^T$;
(f) after 6 iterations, we find $\nu^\star = 3.78885$, so $\lambda = .76393$, $\mathbf{u} = (\,.28832, .27970, -.91577\,)^T$;
(g) after 9 iterations, we find $\nu^\star = -8.47213$, so $\lambda = .38197$, $\mathbf{u} = (\,-.37175, -.60150, -.60150, -.37175\,)^T$;
(h) after 14 iterations, we find $\nu^\star = .66665$, so $\lambda = 2.00003$, $\mathbf{u} = (\,.50001, -.50000, .49999, -.50000\,)^T$.

♠ 10.6.9. (i) Explain how to use the deflation method of Exercise 8.2.51 to find the subdominant eigenvalue of a nonsingular matrix A. (ii) Apply your method to the matrices in Exercise 10.6.1.

Solution:
(i) ■

10.6.10. Suppose that $A\,\mathbf{u}^{(k)} = \mathbf{0}$ in the iterative procedure (10.102). What does this indicate?

Solution: A is a singular matrix and 0 is an eigenvalue. The corresponding eigenvectors are in $\ker A$. In fact, assuming $\mathbf{u}^{(k)} \neq \mathbf{0}$, the iterates $\mathbf{u}^{(0)}, \ldots, \mathbf{u}^{(k)}$ form a Jordan chain for the zero eigenvalue. To find other eigenvalues and eigenvectors, you need to try a different initial vector $\mathbf{u}^{(0)}$.

10.6.11. Apply the $Q\,R$ algorithm to the following symmetric matrices to find their eigenvalues and eigenvectors to 2 decimal places: (a) $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 1 \end{pmatrix}$,

(d) $\begin{pmatrix} 2 & 5 & 0 \\ 5 & 0 & -3 \\ 0 & -3 & 3 \end{pmatrix}$, (e) $\begin{pmatrix} 3 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{pmatrix}$, (f) $\begin{pmatrix} 6 & 1 & -1 & 0 \\ 1 & 8 & 1 & -1 \\ -1 & 1 & 4 & 1 \\ 0 & -1 & 1 & 3 \end{pmatrix}$.

Solution:

(a) eigenvalues: $5, 0$; eigenvectors: $\begin{pmatrix} .4472 \\ .8944 \end{pmatrix}, \begin{pmatrix} .8944 \\ -.4472 \end{pmatrix}$.

(b) eigenvalues: $5.4142, 2.5858$; eigenvectors: $\begin{pmatrix} -.3827 \\ .9239 \end{pmatrix}, \begin{pmatrix} .9239 \\ .3827 \end{pmatrix}$.

(c) eigenvalues: $4.7577, 1.9009, -1.6586$; eigenvectors: $\begin{pmatrix} .2726 \\ .7519 \\ .6003 \end{pmatrix}, \begin{pmatrix} .9454 \\ -.0937 \\ -.3120 \end{pmatrix}, \begin{pmatrix} -.1784 \\ .6526 \\ -.7364 \end{pmatrix}$.

(d) eigenvalues: $7.0988, 2.7191, -4.8180$;

eigenvectors: $\begin{pmatrix} .6205 \\ .6328 \\ -.4632 \end{pmatrix}, \begin{pmatrix} -.5439 \\ -.0782 \\ -.8355 \end{pmatrix}, \begin{pmatrix} .5649 \\ -.7704 \\ -.2956 \end{pmatrix}$.

(e) eigenvalues: $4.6180, 3.6180, 2.3820, 1.3820$;

eigenvectors: $\begin{pmatrix} -.3717 \\ .6015 \\ -.6015 \\ .3717 \end{pmatrix}, \begin{pmatrix} -.6015 \\ .3717 \\ .3717 \\ -.6015 \end{pmatrix}, \begin{pmatrix} -.6015 \\ -.3717 \\ .3717 \\ .6015 \end{pmatrix}, \begin{pmatrix} .3717 \\ .6015 \\ .6015 \\ .3717 \end{pmatrix}$.

(f) eigenvalues: $8.6091, 6.3083, 4.1793, 1.9033$;

eigenvectors: $\begin{pmatrix} -.3182 \\ -.9310 \\ -.1008 \\ .1480 \end{pmatrix}, \begin{pmatrix} .8294 \\ -.2419 \\ -.4976 \\ -.0773 \end{pmatrix}, \begin{pmatrix} .4126 \\ -.1093 \\ .6419 \\ .6370 \end{pmatrix}, \begin{pmatrix} -.2015 \\ .2507 \\ -.5746 \\ .7526 \end{pmatrix}$.

10.6.12. Show that applying the $Q\,R$ algorithm to the matrix $A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 7 & 2 \\ 1 & 2 & 7 \end{pmatrix}$ results in a diagonal matrix with the eigenvalues on the diagonal, but not in decreasing order. Explain.

Solution: eigenvalues: $9, 6, 3$; eigenvectors: $\begin{pmatrix} 0 \\ .7071 \\ .7071 \end{pmatrix}, \begin{pmatrix} .5774 \\ -.5774 \\ .5774 \end{pmatrix}, \begin{pmatrix} .8165 \\ .4082 \\ -.4082 \end{pmatrix}$. The reason is because the eigenvector matrix S^T is not regular.

10.6.13. Apply the $Q\,R$ algorithm to the following non-symmetric matrices to find their eigenvalues to 3 decimal places:

(a) $\begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & 1 & 0 \\ 2 & 0 & -3 \\ 0 & -2 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 2 & 5 & 1 \\ 2 & -1 & 3 \\ 4 & 5 & 3 \end{pmatrix}$, (e) $\begin{pmatrix} 6 & 1 & 7 & 9 \\ 6 & 8 & 14 & 9 \\ 3 & 1 & 4 & 6 \\ 3 & 2 & 5 & 3 \end{pmatrix}$.

Solution:

(a) eigenvalues: $2, 1$; eigenvectors: $\begin{pmatrix} -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) eigenvalues: $1.2087, 5.7913$; eigenvectors: $\begin{pmatrix} -.9669 \\ .2550 \end{pmatrix}, \begin{pmatrix} -.6205 \\ -.7842 \end{pmatrix}$.

(c) eigenvalues: $3.5842, -2.2899, 1.7057$;

$$\text{eigenvectors:} \quad \begin{pmatrix} -.4466 \\ -.7076 \\ 0.5476 \end{pmatrix}, \begin{pmatrix} .1953 \\ -.8380 \\ -.5094 \end{pmatrix}, \begin{pmatrix} .7491 \\ -.2204 \\ .6247 \end{pmatrix}.$$

(*d*) eigenvalues: $7.7474, -.2995, -3.4479$;

$$\text{eigenvectors:} \quad \begin{pmatrix} -.4697 \\ -.3806 \\ -.7966 \end{pmatrix}, \begin{pmatrix} -.7799 \\ .2433 \\ .5767 \end{pmatrix}, \begin{pmatrix} .6487 \\ -.7413 \\ .1724 \end{pmatrix}.$$

(*e*) eigenvalues: $18.3344, 4.2737, 0, -1.6081$;

$$\text{eigenvectors:} \quad \begin{pmatrix} .4136 \\ .8289 \\ .2588 \\ .2734 \end{pmatrix}, \begin{pmatrix} -.4183 \\ .9016 \\ -.0957 \\ .0545 \end{pmatrix}, \begin{pmatrix} -.5774 \\ -.5774 \\ .5774 \\ 0 \end{pmatrix}, \begin{pmatrix} -.2057 \\ .4632 \\ -.6168 \\ .6022 \end{pmatrix}.$$

10.6.14. The matrix $A = \begin{pmatrix} -1 & 2 & 1 \\ -2 & 3 & 1 \\ -2 & 2 & 2 \end{pmatrix}$ has a double eigenvalue of 1, and so our proof of convergence of the QR algorithm doesn't apply. Does the QR algorithm find its eigenvalues?

Solution: Yes. After 10 iterations, one finds

$$R_{10} = \begin{pmatrix} 2.0011 & 1.4154 & 4.8983 \\ 0 & .9999 & -.0004 \\ 0 & 0 & .9995 \end{pmatrix}, \qquad S_{10} = \begin{pmatrix} -.5773 & .4084 & .7071 \\ -.5774 & .4082 & -.7071 \\ -.5774 & -.8165 & .0002 \end{pmatrix},$$

so the diagonal entries of R_{10} give the eigenvalues correct to 3 decimal places, and the columns of S_{10} are similar approximations to the orthonormal eigenvector basis.

10.6.15. Explain why the QR algorithm fails to find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution: It has eigenvalues ± 1, which have the same magnitude. The QR factorization is trivial, with $Q = A$ and $R = \mathrm{I}$. Thus, $RQ = A$, and so nothing happens.

♢ 10.6.16. Prove that all of the matrices A_k defined in (10.103) have the same eigenvalues.

Solution: This follows directly from Exercise 8.2.22.

♢ 10.6.17. Prove that if A is symmetric and tridiagonal, then all matrices A_k appearing in the QR algorithm are also symmetric and tridiagonal.

Solution: ■

10.6.18. Use Householder matrices to convert the following matrices into tridiagonal form:

$$\text{(a)} \begin{pmatrix} 8 & -7 & 2 \\ -7 & 17 & -7 \\ 2 & -7 & 8 \end{pmatrix}, \quad (b) \begin{pmatrix} 5 & 1 & -2 & 1 \\ 1 & 5 & 1 & -2 \\ -2 & 1 & 5 & 1 \\ 1 & -2 & 1 & 5 \end{pmatrix}, \quad (c) \begin{pmatrix} 4 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 3 \end{pmatrix}.$$

Solution:

(a) $H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -.9615 & .2747 \\ 0 & .2747 & .9615 \end{pmatrix}$, $T = HAH = \begin{pmatrix} 8 & 7.2801 & 0 \\ 7.2801 & 20.0189 & 3.5660 \\ 0 & 3.5660 & 4.9811 \end{pmatrix}$..

(*b*) $H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -.4082 & .8165 & -.4082 \\ 0 & .8165 & .5266 & .2367 \\ 0 & -.4082 & .2367 & .8816 \end{pmatrix}$,

$$T_1 = H_1 A H_1 = \begin{pmatrix} 5 & -2.4495 & 0 & 0 \\ -2.4495 & 3.8333 & 1.3865 & .9397 \\ 0 & 1.3865 & 6.2801 & -.9566 \\ 0 & .9397 & -.9566 & 6.8865 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.8278 & -.5610 \\ 0 & 0 & -.5610 & .8278 \end{pmatrix},$$

$$T = H_2 T_1 H_2 = \begin{pmatrix} 5 & -2.4495 & 0 & 0 \\ -2.4495 & 3.8333 & -1.6750 & 0 \\ 0 & -1.6750 & 5.5825 & .0728 \\ 0 & 0 & .0728 & 7.5842 \end{pmatrix}.$$

(c) $$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & .7071 & -.7071 \\ 0 & .7071 & .5 & .5 \\ 0 & -.7071 & .5 & .5 \end{pmatrix},$$

$$T_1 = H_1 A H_1 = \begin{pmatrix} 4 & -1.4142 & 0 & 0 \\ -1.4142 & 2.5 & .1464 & -.8536 \\ 0 & .1464 & 1.0429 & .75 \\ 0 & -.8536 & .75 & 2.4571 \end{pmatrix},$$

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.1691 & .9856 \\ 0 & 0 & .9856 & .1691 \end{pmatrix},$$

$$T = H_2 T_1 H_2 = \begin{pmatrix} 4 & -1.4142 & 0 & 0 \\ -1.4142 & 2.5 & -.8660 & 0 \\ 0 & -.8660 & 2.1667 & .9428 \\ 0 & 0 & .9428 & 1.3333 \end{pmatrix}.$$

♠ 10.6.19. Find the eigenvalues, to 2 decimal places, of the matrices in Exercise 10.6.18 by applying the QR algorithm to the tridiagonal form.

Solution: (a) eigenvalues: $24, 6, 3$. (b) eigenvalues: $7.6180, 7.5414, 5.3820, 1.4586$. (c) eigenvalues: $4.9354, 3.0000, 1.5374, .5272$.

♠ 10.6.20. Use the tridiagonal QR method to find the singular values of the following matrices:

(a) $\begin{pmatrix} 2 & 1 & -1 \\ -2 & 3 & 1 \end{pmatrix}$, $\quad (b)$ $\begin{pmatrix} 2 & 2 & 1 & -1 \\ 1 & -2 & 0 & 1 \end{pmatrix}$, $\quad (c)$ $\begin{pmatrix} 3 & 1 & -1 \\ 1 & -2 & 2 \\ 2 & -1 & 1 \end{pmatrix}$.

Solution:

(a) ■ Omit???

10.6.21. Use Householder matrices to convert the following matrices into upper Hessenberg form:

(a) $\begin{pmatrix} 3 & -1 & 2 \\ 1 & 3 & 4 \\ 2 & -1 & 5 \end{pmatrix}$, $\quad (b)$ $\begin{pmatrix} 3 & 2 & -1 & 1 \\ 2 & 4 & 0 & 1 \\ 0 & 1 & 2 & -6 \\ 1 & 0 & -5 & 1 \end{pmatrix}$, $\quad (c)$ $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 2 & 1 & 1 & -1 \\ -1 & 0 & 1 & 3 \\ 3 & -1 & 1 & 4 \end{pmatrix}$.

Solution:

(a) ■

(b)

(c)

♠ 10.6.22. Find the eigenvalues, to 2 decimal places, of the matrices in Exercise 10.6.21 by applying the QR algorithm to the upper Hessenberg form.

Solution:
(a) eigenvalues: $4.51056, 2.74823, -2.25879$,
(b) eigenvalues: $7., 5.74606, -4.03877, 1.29271$,
(c) eigenvalues: $4.96894, 2.31549, -1.70869, 1.42426$.

10.6.23. Prove that the effect of the first Householder reflection is as given in (10.111).

Solution: First, by Lemma 10.58, $H_1 \mathbf{x}_1 = \mathbf{y}_1$. Furthermore, since the first entry of $\mathbf{u}_1$ is zero, $\mathbf{u}_1^T \mathbf{e}_1 = 0$, and so $H \mathbf{e}_1 = (\mathrm{I} - 2\mathbf{u}_1 \mathbf{u}_1^T)\mathbf{e}_1 = \mathbf{e}_1$. Thus, the first column of $H_1 A$ is

$$H_1(a_{11}\mathbf{e}_1 + \mathbf{x}_1) = a_{11}\mathbf{e}_1 + \mathbf{y}_1 = (\, a_{11}, \pm r, 0, \ldots, 0\,)^T.$$

Finally, again since the first entry of $\mathbf{u}_1$ is zero, the first column of H_1 iis $\mathbf{e}_1$ and so multiplying $H_1 A$ on the right by H_1 doesn't affect its first column. We conclude that the first column of the symmetric matrix $H_1 A H_1$ has the form given in (10.111); symmetry implies that its first row is just the transpose of the first column, which completes the proof. *Q.E.D.*

10.6.24. What is the effect of tridiagonalization on the eigenvectors of the matrix?

Solution: Since $T = H^{-1} A H$ where $H = H_1 H_2 \cdots H_n$ is the product of nthe Househbolder reflections, $A\mathbf{v} = \lambda \mathbf{v}$ if and only if $T\mathbf{w} = \lambda \mathbf{w}$ where $\mathbf{w} = H^{-1}\mathbf{v}$ is the eigenvector of the tridiagonalized matrix. thuls, to recover the eigenvectors of A we need to multiply $\mathbf{v} = H\mathbf{w} = H_1 H_2 \cdots H_n \mathbf{w}$.

♢ 10.6.25. (a) How many arithmetic operations (multiplications/divisions and additions/subtractions) are required to place a generic $n \times n$ matrix into tridiagonal form? (b) How many operations are need to perform one iteration of the QR algorithm on an $n \times n$ tridiagonal matrix? (c) How much faster, on average, is the tridiagonal algorithm than the direct algorithm for finding the eigenvalues of a matrix?