Solutions — Chapter 8

sds 8.1. Simple Dynamical Systems.

8.1.1. Solve the following initial value problems:
(a) $\dfrac{du}{dt} = 5\,u,\ \ u(0) = -3,$ (b) $\dfrac{du}{dt} = 2\,u,\ \ u(1) = 3,$ (c) $\dfrac{du}{dt} = -3\,u,\ \ u(-1) = 1.$

Solution: (a) $u(t) = -3\,e^{5\,t}$, (b) $u(t) = 3\,e^{2(t-1)}$, (c) $u(t) = e^{-3(t+1)}$.

8.1.2. Suppose a radioactive material has a half-life of 100 years. What is the decay rate γ? Starting with an initial sample of 100 grams, how much will be left after 10 years? 100 years? 1,000 years?

Solution: $\gamma = \log 2/100 \approx 0.0069$. After 10 years: 93.3033 gram; after 100 years 50 gram; after 1000 years 0.0977 gram.

8.1.3. Carbon-14 has a half-life of 5730 years. Human skeletal fragments discovered in a cave are analyzed and found to have only 6.24% of the carbon-14 that living tissue would have. How old are the remains?

Solution: Solve $e^{-(\log 2)t/5730} = .0624$ for $t = -5730 \log .0624/\log 2 = 22{,}933$ years.

8.1.4. Prove that if $t^\star$ is the half-life of a radioactive material, then $u(n\,t^\star) = 2^{-n}\,u(0)$. Explain the meaning of this equation in your own words.

Solution: By (8.6), $u(t) = u(0)\,e^{-(\log 2)t/t^\star} = u(0)\left(\frac{1}{2}\right)^{t/t^\star} = 2^{-n}\,u(0)$ when $t = n\,t^\star$. After every time period of duration $t^\star$, the amount of material is cut in half.

8.1.5. A bacteria colony grows according to the equation $du/dt = 1.3\,u$. How long until the colony doubles? quadruples? If the initial population is 2, how long until the population reaches 2 million?

Solution: $u(t) = u(0)\,e^{1.3\,t}$. To double, we need $e^{1.3\,t} = 2$, so $t = \log 2/1.3 = 0.5332$. To quadruple takes twice as long, $t = 1.0664$. To reach 2 million needs $t = \log 10^6/1.3 = 10.6273$.

8.1.6. Deer in Northern Minnesota reproduce according to the linear differential equation $\dfrac{du}{dt} = .27\,u$ where t is measured in years. If the initial population is $u(0) = 5{,}000$ and the environment can sustain at most $1{,}000{,}000$ deer, how long until the deer run out of resources?

Solution: The solution is $u(t) = u(0)\,e^{.27\,t}$. For the given initial conditions, $u(t) = 1{,}000{,}000$ when $t = \log(1000000/5000)/.27 = 19.6234$ years.

◇ 8.1.7. Consider the inhomogeneous differential equation $\dfrac{du}{dt} = a\,u + b$, where a, b are constants. (a) Show that $u_\star = -b/a$ is a constant equilibrium solution. (b) Solve the differential equation. *Hint*: Look at the differential equation satisfied by $v = u - u_\star$. (c) Discuss the stability of the equilibrium solution $u_\star$.

Solution:

(a) If $u(t) \equiv u_\star = -\dfrac{b}{a}$, then $\dfrac{du}{dt} = 0 = a\,u + b$, hence it is a solution.

(b) $v = u - u_\star$ satisfies $\dfrac{dv}{dt} = a\,v$, so $v(t) = c e^{a\,t}$ and so $u(t) = c\,e^{a\,t} - \dfrac{b}{a}$.

(c) The equilibrium solution is asymptotically stable if and only if $a < 0$ and is stable if $a = 0$.

8.1.8. Use the method of Exercise 8.1.7 to solve the following initial value problems:
(a) $\dfrac{du}{dt} = 2\,u - 1,\ \ u(0) = 1$, (b) $\dfrac{du}{dt} = 5\,u + 15,\ \ u(1) = -3$, (c) $\dfrac{du}{dt} = -3\,u + 6,\ \ u(2) = -1$.

Solution: (a) $u(t) = \frac{1}{2} + \frac{1}{2}\,e^{2\,t}$, (b) $u(t) = -3$, (c) $u(t) = 2 - 3\,e^{-3(t-2)}$.

8.1.9. The radioactive waste from a nuclear reactor has a half-life of 1000 years. Waste is continually produced at the rate of 5 tons per year and stored in a dump site. (a) Set up an inhomogeneous differential equation, of the form in Exercise 8.1.7, to model the amount of radioactive waste. (b) Determine whether the amount of radioactive material at the dump increases indefinitely, decreases to zero, or eventually stabilizes at some fixed amount. (c) Starting with a brand new site, how long until the dump contain 100 tons of radioactive material?

Solution: (a) $\dfrac{du}{dt} = -\dfrac{\log 2}{1000}\,u + 5 \approx -.000693\,u + 5$. (b) Stabilizes at the equilibrium solution $u_\star = 5000/\log 2 \approx 721$ tons. (c) The solution is $u(t) = \dfrac{5000}{\log 2}\left[\,1 - \exp\left(-\,\frac{\log 2}{1000}\,t\right)\right]$ which equals 100 when $t = -\dfrac{1000}{\log 2}\log\left(1 - 100\,\dfrac{\log 2}{5000}\right) \approx 20.14$years.

♡ 8.1.10. Suppose that hunters are allowed to shoot a fixed number of the Northern Minnesota deer in Exercise 8.1.6 each year. (a) Explain why the population model takes the form $\dfrac{du}{dt} = .27\,u - b$, where b is the number killed yearly. (Ignore the season aspects of hunting.) (b) If $b = 1{,}000$, how long until the deer run out of resources? *Hint*: See Exercise 8.1.7. (c) What is the maximal rate at which deer can be hunted without causing their extinction?

Solution:

(a) The rate of growth remains proportional to the population, while hunting decreases the population by a fixed amount. (This assumes hunting is done continually throughout the year, which is not what happens in real life.)

(b) The solution is $u(t) = \left(5000 - \frac{1000}{.27}\right)e^{.27\,t} + \frac{1000}{.27}$. Solving $u(t) = 100000$ gives $t = \dfrac{1}{.27}\log\dfrac{1000000 - 1000/.27}{5000 - 1000/.27} = 24.6094$ years.

(c) We need the equilibrium $u^\star = b/.27$ to be less than the initial population, so $b < 1350$ deer.

◇ 8.1.11. (a) Prove that if $u_1(t)$ and $u_2(t)$ are any two distinct solutions to $\dfrac{du}{dt} = a\,u$ with $a > 0$, then $|\,u_1(t) - u_2(t)\,| \to \infty$ as $t \to \infty$. (b) If $a = .02$ and $u_1(0) = .1$, $u_2(0) = .05$, how long do you have to wait until $|\,u_1(t) - u_2(t)\,| > 1{,}000$?

Solution:

(a) $|\,u_1(t) - u_2(t)\,| = e^{a\,t}\,|\,u_1(0) - u_2(0)\,| \to \infty$ when $a > 0$.

(b) $t = \log(1000/.05)/.02 = 495.17$.

8.1.12. (a) Write down the exact solution to the initial value problem $\dfrac{du}{dt} = \dfrac{2}{7}\,u,\ \ u(0) = \dfrac{1}{3}$.

(b) Suppose you make the approximation $u(0) = .3333$. At what point does your solution differ from the true solution by 1 unit? by 1000? (c) Answer the same question if you also approximate the coefficient in the differential equation by $\dfrac{du}{dt} = .2857\,u$.

Solution:

(a) $u(t) = \frac{1}{3}\, e^{2\,t/7}$.

(b) One unit: $t = \log\big[\,1/(1/3 - .3333)\,\big]/(2/7) = 36.0813$;
1000 units: $t = \log\big[\,1000/(1/3 - .3333)\,\big]/(2/7) = 60.2585$

(c) One unit: $t \approx 30.2328$ solves $\frac{1}{3}\, e^{2\,t/7} - .3333\, e^{.2857\,t} = 1$. (Use a numerical equation solver.) 1000 units: $t \approx 52.7548$ solves $\frac{1}{3}\, e^{2\,t/7} - .3333\, e^{.2857\,t} = 1000$.

♢ 8.1.13. Let a be complex. Prove that $u(t) = c\,e^{a\,t}$ is the (complex) solution to our scalar ordinary differential equation (8.1). Describe the asymptotic behavior of the solution as $t \to \infty$, and the stability properties of the zero equilibrium solution.

Solution: The solution is still valid as a complex solution. If $\operatorname{Re} a > 0$, then $u(t) \to \infty$ as $t \to \infty$, and the origin is an unstable equilibrium . If $\operatorname{Re} a = 0$, then $u(t)$ remains bounded $t \to \infty$, and the origin is a stable equilibrium. If $\operatorname{Re} a < 0$, then $u(t) \to 0$ as $t \to \infty$, and the origin is an asymptotically stable equilibrium.

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8.2. Eigenvalues and Eigenvectors.

8.2.1. Find the eigenvalues and eigenvectors of the following matrices:

$$(a)\ \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 1 & -\frac{2}{3} \\ \frac{1}{2} & -\frac{1}{6} \end{pmatrix}, \quad (c)\ \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}, \quad (d)\ \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix},$$

$$(e)\ \begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}, \quad (f)\ \begin{pmatrix} -1 & -1 & 4 \\ 1 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix}, \quad (g)\ \begin{pmatrix} 1 & -3 & 11 \\ 2 & -6 & 16 \\ 1 & -3 & 7 \end{pmatrix}, \quad (h)\ \begin{pmatrix} 2 & -1 & -1 \\ -2 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

$$(i)\ \begin{pmatrix} -4 & -4 & 2 \\ 3 & 4 & -1 \\ -3 & -2 & 3 \end{pmatrix}, \quad (j)\ \begin{pmatrix} 3 & 4 & 0 & 0 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 4 & 5 \end{pmatrix}, \quad (k)\ \begin{pmatrix} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

Solution:

(a) Eigenvalues: $3, -1$; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(b) Eigenvalues: $\frac{1}{2}, \frac{1}{3}$; eigenvectors: $\begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Eigenvalues: 2; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(d) Eigenvalues: $1 + \mathrm{i}\sqrt{2}, 1 - \mathrm{i}\sqrt{2}$; eigenvectors: $\begin{pmatrix} -\mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}$.

(e) Eigenvalues: $4, 3, 1$; eigenvectors: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

(f) Eigenvalues: $1, \sqrt{6}, -\sqrt{6}$; eigenvectors: $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1+\frac{\sqrt{3}}{\sqrt{2}} \\ 2+\frac{\sqrt{3}}{\sqrt{2}} \\ 1 \end{pmatrix}, \begin{pmatrix} -1-\frac{\sqrt{3}}{\sqrt{2}} \\ 2-\frac{\sqrt{3}}{\sqrt{2}} \\ 1 \end{pmatrix}$.

(g) Eigenvalues: $0, 1+\mathrm{i}, 1-\mathrm{i}$; eigenvectors: $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3-2\,\mathrm{i} \\ 3-\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} 3+2\,\mathrm{i} \\ 3+\mathrm{i} \\ 1 \end{pmatrix}$.

(h) Eigenvalues: $2, 0$; eigenvectors: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$

(i) -1 simple eigenvalue, eigenvector $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; 2 double eigenvalue, eigenvectors $\begin{pmatrix} \frac{1}{3} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{pmatrix}$

(j) Two double eigenvalues: -1, eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}$ and 7, eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}$.

(k) Eigenvalues: $1, 2, 3, 4$; eigenvectors: $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

8.2.2. (a) Find the eigenvalues of the rotation matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. For what values of θ are the eigenvalues real? (b) Explain why your answer gives an immediate solution to Exercise 1.5.7c.

Solution: (a) The eigenvalues are $\cos\theta \pm \mathrm{i}\sin\theta = e^{\pm\mathrm{i}\theta}$ with eigenvectors $\begin{pmatrix} 1 \\ \mp\mathrm{i} \end{pmatrix}$. They are real only for $\theta = 0$ and π. (b) Because $R_\theta - a\,\mathrm{I}$ has an inverse if and only if a is not an eigenvalue.

8.2.3. Answer Exercise 8.2.2a for the reflection matrix $F_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Solution: The eigenvalues are ± 1 with eigenvectors $(\sin\theta, \cos\theta \mp 1)^T$.

8.2.4. Write down (a) a 2×2 matrix that has 0 as one of its eigenvalues and $(1,2)^T$ as a corresponding eigenvector; (b) a 3×3 matrix that has $(1,2,3)^T$ as an eigenvector for the eigenvalue -1.

Solution: (a) O, and (b) $-\mathrm{I}$, are trivial examples.

8.2.5. (a) Write out the characteristic equation for the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & \beta & \gamma \end{pmatrix}$.

(b) Show that, given any 3 numbers a, b, and c, there is a 3×3 matrix with characteristic equation $-\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

Solution:
(a) The characteristic equation is $-\lambda^3 + a\lambda^2 + b\lambda + c = 0$.
(b) Use the matrix in part (a).

8.2.6. Find the eigenvalues and eigenvectors of the cross product matrix $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$.

Solution: The eigenvalues are $0, \pm\mathrm{i}\sqrt{a^2+b^2+c^2}$. If $a = b = c = 0$ then $A = \mathrm{O}$ and all vectors are eigenvectors. Otherwise, the eigenvectors are $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\begin{pmatrix} b \\ -a \\ 0 \end{pmatrix} \mp \dfrac{\mathrm{i}}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$.

8.2.7. Find all eigenvalues and eigenvectors of the following complex matrices:
(a) $\begin{pmatrix} \mathrm{i} & 1 \\ 0 & -1+\mathrm{i} \end{pmatrix}$, (b) $\begin{pmatrix} 2 & \mathrm{i} \\ -\mathrm{i} & -2 \end{pmatrix}$, (c) $\begin{pmatrix} \mathrm{i}-2 & \mathrm{i}+1 \\ \mathrm{i}+2 & \mathrm{i}-1 \end{pmatrix}$, (d) $\begin{pmatrix} 1+\mathrm{i} & -1-\mathrm{i} & 1-\mathrm{i} \\ 2 & -2-\mathrm{i} & 2-2\mathrm{i} \\ -1 & 1+\mathrm{i} & -1+2\mathrm{i} \end{pmatrix}$.

Solution:

(a) Eigenvalues: $\mathrm{i}, -1+\mathrm{i}$; eigenvectors: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) Eigenvalues: $\pm\sqrt{5}$; eigenvectors: $\begin{pmatrix} \mathrm{i}(2 \pm \sqrt{5}) \\ 1 \end{pmatrix}$.

(c) Eigenvalues: $-3, 2\mathrm{i}$; eigenvectors: $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \frac{3}{5}+\frac{1}{5}\mathrm{i} \\ 1 \end{pmatrix}$.

(d) -2 simple, eigenvector $\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$; i double, eigenvectors $\begin{pmatrix} -1+\mathrm{i} \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1+\mathrm{i} \\ 1 \\ 0 \end{pmatrix}$.

8.2.8. Find all eigenvalues and eigenvectors of (a) the $n \times n$ zero matrix O; (b) the $n \times n$ identity matrix I.

Solution:

(a) Since $\mathrm{O}\,\mathbf{v} = \mathbf{0} = 0\,\mathbf{v}$, we conclude that 0 is the only eigenvalue; all nonzero vectors are eigenvectors.

(b) Since $\mathrm{I}\,\mathbf{v} = \mathbf{v} = 1\,\mathbf{v}$, we conclude that 1 is the only eigenvalue; all nonzero vectors are eigenvectors.

8.2.9. Find the eigenvalues and eigenvectors of an $n \times n$ matrix with every entry equal to 1. *Hint*: Try with $n = 2, 3$, and then generalize.

Solution: For $n = 2$, the eigenvalues are $0, 2$, and the eigenvectors are $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $n = 3$, the eigenvalues are $0, 0, 3$, and the eigenvectors are $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. In general, the eigenvalues are 0, with multiplicity $n-1$, and n, which is simple. The eigenvectors corresponding to the 0 eigenvalues are of the form $(v_1, v_2, \ldots, v_n)^T$ where $v_1 = -1$ and $v_j = 1$ for $j = 2, \ldots, n$. The eigenvector corresponding to n is $(1, 1, \ldots, 1)^T$.

◇ 8.2.10. Let A be a given square matrix. (a) Explain in detail why any nonzero scalar multiple of an eigenvector of A is also an eigenvector. (b) Show that any nonzero linear combination of two eigenvectors $\mathbf{v}, \mathbf{w}$ corresponding to the *same* eigenvalue is also an eigenvector. (c) Prove that a linear combination $c\mathbf{v} + d\mathbf{w}$ with $c, d \neq 0$ of two eigenvectors corresponding to *different* eigenvalues is never an eigenvector.

Solution:

(a) If $A\mathbf{v} = \lambda\mathbf{v}$, then $A(c\mathbf{v}) = cA\mathbf{v} = c\lambda\mathbf{v} = \lambda(c\mathbf{v})$ and so $c\mathbf{v}$ satisfies the eigenvector equation with eigenvalue λ. Moreover, since $\mathbf{v} \neq \mathbf{0}$, also $c\mathbf{v} \neq \mathbf{0}$ for $c \neq 0$, and so $c\mathbf{v}$ is a bona fide eigenvector.

(b) If $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \lambda\mathbf{w}$, then $A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\lambda\mathbf{w} = \lambda(c\mathbf{v} + d\mathbf{w})$.

(c) Suppose $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \mu\mathbf{w}$. Then $\mathbf{v}$ and $\mathbf{w}$ must be linearly independent as otherwise they would be scalar multiples of each other and hence have the same eigenvalue. Thus, $A(c\mathbf{v} + d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\lambda\mathbf{v} + d\mu\mathbf{w} = \nu(c\mathbf{v} + d\mathbf{w})$ if and only if $c\lambda = c\nu$ and $d\mu = d\nu$, which, when $\lambda \neq \mu$, is only possible if either $c = 0$ or $d = 0$.

8.2.11. *True or false*: If $\mathbf{v}$ is a real eigenvector of a real matrix A, then a nonzero complex multiple $\mathbf{w} = c\,\mathbf{v}$ for $c \in \mathbb{C}$ is a complex eigenvector of A.

Solution: True — by the same computation as in Exercise 8.2.10(a), $c\,\mathbf{v}$ is an eigenvector for the same (real) eigenvalue λ.

♢ 8.2.12. Define the *shift map* $S: \mathbb{C}^n \to \mathbb{C}^n$ by $S\left(v_1, v_2, \ldots, v_{n-1}, v_n\right)^T = \left(v_2, v_3, \ldots, v_n, v_1\right)^T$.
(*a*) Prove that S is a linear map, and write down its matrix representation A.
(*b*) Prove that A is an orthogonal matrix.
(*c*) Prove that the sampled exponential vectors $\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_{n-1}$ in (5.84) form an eigenvector basis of A. What are the eigenvalues?

Solution:

(*a*) $$A = \begin{pmatrix} 0 & 1 & 0 & & & & 0 & 0 \\ & 0 & 1 & 0 & & & & 0 \\ & & 0 & 1 & 0 & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & & 0 & 1 & 0 \\ 0 & & & & & & 0 & 1 \\ 1 & 0 & & & & & & 0 \end{pmatrix}.$$

(*b*) $A^T A = \mathrm{I}$ by direct computation, or, equivalently, the columns of A are the standard orthonormal basis vectors $\mathbf{e}_n, \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}$, written in a slightly different order.

(*c*) Since
$$\boldsymbol{\omega}_k = \left(1, e^{2k\pi\,\mathrm{i}/n}, e^{4k\pi\,\mathrm{i}/n}, \ldots, e^{2(n-1)k\pi\,\mathrm{i}/n}\right)^T,$$
$$S\boldsymbol{\omega}_k = \left(e^{2k\pi\,\mathrm{i}/n}, e^{4k\pi\,\mathrm{i}/n}, \ldots, e^{2(n-1)k\pi\,\mathrm{i}/n,1}\right)^T = e^{2k\pi\,\mathrm{i}/n}\,\boldsymbol{\omega}_k,$$
and so $\boldsymbol{\omega}_k$ is an eigenvector with corresponding eigenvalue $e^{2k\pi\,\mathrm{i}/n}$.

8.2.13. (*a*) Compute the eigenvalues and corresponding eigenvectors of $A = \begin{pmatrix} 1 & 4 & 4 \\ 3 & -1 & 0 \\ 0 & 2 & 3 \end{pmatrix}$.
(*b*) Compute the trace of A and check that it equals the sum of the eigenvalues. (*c*) Find the determinant of A, and check that it is equal to to the product of the eigenvalues.

Solution: (*a*) Eigenvalues: $-3, 1, 5$; eigenvectors: $(2,-3,1)^T, \left(-\frac{2}{3}, -1, 1\right)^T, (2,1,1)^T$.
(*b*) $\operatorname{tr} A = 3 = -3 + 1 + 5$ (*c*) $\det A = -15 = (-3)\cdot 1 \cdot 5$.

8.2.14. Verify the trace and determinant formulae (8.24–25) for the matrices in Exercise 8.2.1.

Solution:
(*a*) $\operatorname{tr} A = 2 = 3 + (-1)$; $\det A = -3 = 3 \cdot (-1)$.
(*b*) $\operatorname{tr} A = \frac{5}{6} = \frac{1}{2} + \frac{1}{3}$; $\det A = \frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3}$.
(*c*) $\operatorname{tr} A = 4 = 2 + 2$; $\det A = 4 = 2 \cdot 2$.
(*d*) $\operatorname{tr} A = 2 = (1 + \mathrm{i}\sqrt{2}) + (1 - \mathrm{i}\sqrt{2})$; $\det A = 3 = (1 + \mathrm{i}\sqrt{2}) \cdot (1 - \mathrm{i}\sqrt{2})$.
(*e*) $\operatorname{tr} A = 8 = 4 + 3 + 1$; $\det A = 12 = 4 \cdot 3 \cdot 1$.
(*f*) $\operatorname{tr} A = 1 = 1 + \sqrt{6} + (-\sqrt{6})$; $\det A = -6 = 1 \cdot \sqrt{6} \cdot (-\sqrt{6})$.
(*g*) $\operatorname{tr} A = 2 = 0 + (1 + \mathrm{i}) + (1 - \mathrm{i})$; $\det A = 0 = 0 \cdot (1 + \mathrm{i}\sqrt{2}) \cdot (1 - \mathrm{i}\sqrt{2})$.
(*h*) $\operatorname{tr} A = 4 = 2 + 2 + 0$; $\det A = 0 = 2 \cdot 2 \cdot 0$.
(*i*) $\operatorname{tr} A = 3 = (-1) + 2 + 2$; $\det A = -4 = (-1) \cdot 2 \cdot 2$.
(*j*) $\operatorname{tr} A = 12 = (-1) + (-1) + 7 + 7$; $\det A = 49 = (-1) \cdot (-1) \cdot 7 \cdot 7$.

(k) $\operatorname{tr} A = 10 = 1 + 2 + 3 + 4$; $\det A = 24 = 1 \cdot 2 \cdot 3 \cdot 4$.

8.2.15. (a) Find the explicit formula for the characteristic polynomial $\det(A - \lambda \mathrm{I}) = -\lambda^3 + a\lambda^2 - b\lambda + c$ of a general 3×3 matrix. Verify that $a = \operatorname{tr} A$, $c = \det A$. What is the formula for b? (b) Prove that if A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then $a = \operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3$, $b = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$, $c = \det A = \lambda_1 \lambda_2 \lambda_3$.

Solution:

(a) $a = a_{11} + a_{22} + a_{33} = \operatorname{tr} A$, $b = a_{11} a_{22} - a_{12} a_{21} + a_{11} a_{33} - a_{13} a_{31} + a_{22} a_{33} - a_{23} a_{32}$, $c = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} = \det A$

(b) When the factored form of the characteristic polynomial is multiplied out, we obtain $-(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = -\lambda^3 + (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 - (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)\lambda - \lambda_1 \lambda_2 \lambda_3$, giving the eigenvalue formulas for a, b, c.

8.2.16. Prove that the eigenvalues of an upper triangular (or lower triangular) matrix are its diagonal entries.

Solution: If U is upper triangular, so is $U - \lambda \mathrm{I}$, and hence $p(\lambda) = \det(U - \lambda \mathrm{I})$ is the product of the diagonal entries, so $p(\lambda) = \prod (u_{ii} - \lambda)$, and so the roots of the characteristic equation are the diagonal entries $u_{11}, \ldots, u_{nn}$.

♢ 8.2.17. Let J_a be the $n \times n$ Jordan block matrix (8.22). Prove that its only eigenvalue is $\lambda = a$ and the only eigenvectors are the nonzero scalar multiples of the standard basis vector $\mathbf{e}_1$.

Solution: Since $J_a - \lambda \mathrm{I}$ is an upper triangular matrix with $\lambda - a$ on the diagonal, its determinant is $\det(J_a - \lambda \mathrm{I}) = (a - \lambda)^n$ and hence its only eigenvalue is $\lambda = a$, of multiplicity n. (Or use Exercise 8.2.16.) Moreover, $(J_a - a\,\mathrm{I})\mathbf{v} = (v_2, v_3, \ldots, v_n, 0)^T = \mathbf{0}$ if and only if $\mathbf{v} = c\,\mathbf{e}_1$.

♢ 8.2.18. Suppose that λ is an eigenvalue of A. (a) Prove that $c\lambda$ is an eigenvalue of the scalar multiple cA. (b) Prove that $\lambda + d$ is an eigenvalue of $A + d\,\mathrm{I}$. (c) More generally, $c\lambda + d$ is an eigenvalue of $B = c\,A + d\,\mathrm{I}$ for scalars c, d.

Solution: parts (a), (b) are special cases of part (c): If $A\mathbf{v} = \lambda\,\mathbf{v}$ then $B\mathbf{v} = (cA + d\,\mathrm{I})\mathbf{v} = (c\,\lambda + d)\,\mathbf{v}$.

8.2.19. Show that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

Solution: If $A\mathbf{v} = \lambda\mathbf{v}$ then $A^2\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$, and hence $\mathbf{v}$ is also an eigenvector of A^2 with eigenvalue λ^2.

8.2.20. *True or false*: (a) If λ is an eigenvalue of both A and B, then it is an eigenvalue of the sum $A + B$. (b) If $\mathbf{v}$ is an eigenvector of both A and B, then it is an eigenvector of $A + B$.

Solution: (a) False. For example, 0 is an eigenvalue of both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, but the eigenvalues of $A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ are $\pm\,\mathrm{i}$. (b) True. If $A\mathbf{v} = \lambda\,\mathbf{v}$ and $B\,\mathbf{v} = \mu\,\mathbf{v}$, then $(A+B)\mathbf{v} = (\lambda + \mu)\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda + \mu$.

8.2.21. *True or false*: If λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of the matrix product $C = AB$.

Solution: False in general, but true if the eigenvectors coincide: If $A\mathbf{v} = \lambda\mathbf{v}$ and $B\,\mathbf{v} = \mu\,\mathbf{v}$, then $A\,B\,\mathbf{v} = (\lambda\,\mu)\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector with eigenvalue $\lambda\,\mu$.

♢ 8.2.22. Let A and B be $n \times n$ matrices. Prove that the matrix products $A\,B$ and $B\,A$ have the same eigenvalues.

Solution: If $A\,B\,\mathbf{v} = \lambda\mathbf{v}$, then $B\,A\,\mathbf{w} = \lambda\,\mathbf{w}$, where $\mathbf{w} = B\,\mathbf{v}$. Thus, as long as $\mathbf{w} \neq \mathbf{0}$, it is an eigenvector of BA with eigenvalue λ. However, if $\mathbf{w} = \mathbf{0}$, then $AB\mathbf{v} = \mathbf{0}$, and so the eigenvalue is $\lambda = 0$, which implies that AB is singular. But then so is BA, which also has 0 as an eigenvalue. Thus every eigenvalue of AB is an eigenvalue of BA. The converse follows by the same reasoning. *Note*: This does not imply their null eigenspaces or kernels have the same dimension; compare Exercise 1.8.18. In anticipation of Section 8.6, even though AB and BA have the same eigenvalues, they may have different Jordan canonical forms.

◇ 8.2.23. (*a*) Prove that if $\lambda \neq 0$ is a nonzero eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}. (*b*) What happens if A has 0 as an eigenvalue?

Solution: (a) Starting with $A\mathbf{v} = \lambda\mathbf{v}$, multiply both sides by A^{-1} and divide by λ to obtain $A^{-1}\mathbf{v} = (1/\lambda)\,\mathbf{v}$. Therefore, $\mathbf{v}$ is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.
(*b*) If 0 is an eigenvalue, then A is not invertible.

◇ 8.2.24. (*a*) Prove that if $\det A > 1$ then A has at least one eigenvalue with $|\lambda| > 1$. (*b*) If $|\det A| < 1$, are all eigenvalues $|\lambda| < 1$? Prove or find a counter-example.

Solution: (a) If all $|\lambda_j| \leq 1$ then so is their product $1 \geq |\lambda_1 \ldots \lambda_n| = |\det A|$, which is a contradiction. (*b*) False. $A = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ has eigenvalues $2, \frac{1}{3}$ while $\det A = \frac{2}{3}$.

8.2.25. Prove that A is a singular matrix if and only if 0 is an eigenvalue.

Solution: Recall that A is singular if and only if $\ker A \neq \{\mathbf{0}\}$. Any $\mathbf{v} \in \ker A$ satisfies $A\mathbf{v} = \mathbf{0} = 0\mathbf{v}$. Thus $\ker A$ is nonzero if and only if A has a null eigenvector.

8.2.26. Prove that *every* nonzero vector $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^n$ is an eigenvector of A if and only if A is a scalar multiple of the identity matrix.

Solution: Let $\mathbf{v}, \mathbf{w}$ be any two linearly independent vectors. Then $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$ for some λ, μ. But $\mathbf{v} + \mathbf{w}$ is an eigenvector if and only if $A(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \mu\mathbf{w} = \nu(\mathbf{v} + \mathbf{w})$, which requires $\lambda = \mu = \nu$. Thus, $A\mathbf{v} = \lambda\mathbf{v}$ for every $\mathbf{v}$, which implies $A = \lambda\,\mathrm{I}$.

8.2.27. How many unit eigenvectors correspond to a given eigenvalue of a matrix?

Solution: If λ is a simple real eigenvalue, then there are two real unit eigenvectors: $\mathbf{u}$ and $-\mathbf{u}$. For a complex eigenvalue, if $\mathbf{u}$ is a unit complex eigenvector, so is $e^{\mathrm{i}\theta}\mathbf{u}$, and so there are infinitely many complex unit eigenvectors. (The same holds for a real eigenvalue if we also allow complex eigenvectors.) If λ is a multiple real eigenvalue, with eigenspace of dimension greater than 1, then there are infinitely many unit real eigenvectors in the eigenspace.

8.2.28. *True or false*: (*a*) Performing an elementary row operation of type #1 does not change the eigenvalues of a matrix. (*b*) Interchanging two rows of a matrix changes the sign of its eigenvalues. (*c*) Multiplying one row of a matrix by a scalar multiplies one of its eigenvalues by the same scalar.

Solution: All false. Simple 2×2 examples suffice to disprove them.

8.2.29. (*a*) *True or false*: If $\lambda_1, \mathbf{v}_1$ and $\lambda_2, \mathbf{v}_2$ solve the eigenvalue equation (8.12) for a given matrix A, so does $\lambda_1 + \lambda_2$, $\mathbf{v}_1 + \mathbf{v}_2$. (*b*) Explain what this has to do with linearity.

Solution: False. The eigenvalue equation $A\mathbf{v} = \lambda\mathbf{v}$ is *not* linear in the eigenvalue and eigenvector since $A(\mathbf{v}_1 + \mathbf{v}_2) \neq (\lambda_1 + \lambda_2)(\mathbf{v}_1 + \mathbf{v}_2)$ in general.

8.2.30. An *elementary reflection matrix* has the form $Q = \mathrm{I} - 2\mathbf{u}\mathbf{u}^T$, where $\mathbf{u} \in \mathbb{R}^n$ is a unit vector. (*a*) Find the eigenvalues and eigenvectors for the elementary reflection matrices corresponding to the following unit vectors: (i) $\begin{pmatrix}1\\0\end{pmatrix}$, (ii) $\begin{pmatrix}\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$, (iii) $\begin{pmatrix}0\\1\\0\end{pmatrix}$, (iv) $\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix}$.

(b) What are the eigenvalues and eigenvectors of a general elementary reflection matrix?

Solution:

(a) (i) $Q = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. Eigenvalue -1 has eigenvector: $\begin{pmatrix}1\\0\end{pmatrix}$; Eigenvalue 1 has eigenvector: $\begin{pmatrix}0\\1\end{pmatrix}$. (ii) $Q = \begin{pmatrix}\frac{7}{25} & -\frac{24}{25}\\ -\frac{24}{25} & -\frac{7}{25}\end{pmatrix}$. Eigenvalue -1 has eigenvector: $\begin{pmatrix}\frac{3}{5}\\ \frac{4}{5}\end{pmatrix}$; Eigenvalue 1 has eigenvector: $\begin{pmatrix}\frac{4}{5}\\ -\frac{3}{5}\end{pmatrix}$. (iii) $Q = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Eigenvalue -1 has eigenvector: $\begin{pmatrix}0\\1\\0\end{pmatrix}$; Eigenvalue 1 has eigenvectors: $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$. (iv) $Q = \begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$. Eigenvalue 1 has eigenvector: $\begin{pmatrix}-1\\0\\1\end{pmatrix}$; Eigenvalue -1 has eigenvectors: $\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$.

(b) $\mathbf{u}$ is an eigenvector with eigenvalue -1. All vectors orthogonal to $\mathbf{u}$ are eigenvectors with eigenvalue $+1$.

◇ 8.2.31. Let A and B be similar matrices, so $B = S^{-1} A S$ for some nonsingular matrix S. (a) Prove that A and B have the same characteristic polynomial: $p_B(\lambda) = p_A(\lambda)$. (b) Explain why similar matrices have the same eigenvalues. (c) Do they have the same eigenvectors? If not, how are their eigenvectors related? (d) Prove that the converse is not true by showing that $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ and $\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$ have the same eigenvalues, but are *not* similar.

Solution:

(a)
$$\begin{aligned}\det(B - \lambda\,\mathrm{I}) &= \det(S^{-1} A S - \lambda\,\mathrm{I}) = \det\big[\,S^{-1}(A - \lambda\,\mathrm{I})S\,\big]\\ &= \det S^{-1}\det(A - \lambda\,\mathrm{I})\det S = \det(A - \lambda\,\mathrm{I}).\end{aligned}$$

(b) The eigenvalues are the roots of the common characteristic equation.

(c) Not usually. If $\mathbf{w}$ is an eigenvector of B, then $\mathbf{v} = S\mathbf{w}$ is an eigenvector of A and conversely.

(d) Both have 2 as a double eigenvalue. Suppose $\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix} = S^{-1}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} S$, or, equivalently, $S\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} S$ for some $S = \begin{pmatrix}x & y\\ z & w\end{pmatrix}$. Then, equating entries, we must have $x - y = 2x, x + 3y = 0, z - w = 0, z + 3w = 2w$ which implies $x = y = z = w = 0$, and so $S = \mathrm{O}$, which is not invertible.

8.2.32. Let A be a nonsingular $n \times n$ matrix with characteristic polynomial $p_A(\lambda)$. (a) Explain how to construct the characteristic polynomial $p_{A^{-1}}(\lambda)$ of its inverse directly from $p_A(\lambda)$. (b) Check your result when $A = (i)$ $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$, (ii) $\begin{pmatrix}1 & 4 & 4\\ -2 & -1 & 0\\ 0 & 2 & 3\end{pmatrix}$.

Solution:

(a)
$$p_{A^{-1}}(\lambda) = \det(A^{-1} - \lambda\,\mathrm{I}) = \det \lambda\, A^{-1}\left(\frac{1}{\lambda}\,\mathrm{I} - A\right) = \frac{(-\lambda)^n}{\det A}\, p_A\left(\frac{1}{\lambda}\right).$$

Or, equivalently, if
$$p_A(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0,$$
then, since $c_0 = \det A \neq 0$,
$$p_{A^{-1}}(\lambda) = (-1)^n \left[\lambda^n + \frac{c_1}{c_0}\lambda^{n-1} + \cdots + \frac{c_{n-1}}{c_0}\lambda\right] + \frac{1}{c_0}.$$

(b) (i) $A^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$. Then $p_A(\lambda) = \lambda^2 - 5\lambda - 2$, while
$$p_{A^{-1}}(\lambda) = \lambda^2 + \tfrac{5}{2}\lambda - \tfrac{1}{2} = \frac{\lambda^2}{2}\left(-\frac{2}{\lambda^2} - \frac{5}{\lambda} + 1\right).$$

(ii) $A^{-1} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} & \frac{4}{5} \\ \frac{6}{5} & \frac{3}{5} & -\frac{8}{5} \\ -\frac{4}{5} & -\frac{2}{5} & \frac{7}{5} \end{pmatrix}$. Then $p_A(\lambda) = -\lambda^3 + 3\lambda^2 - 7\lambda + 5$, while
$$p_{A^{-1}}(\lambda) = -\lambda^3 + \tfrac{7}{5}\lambda^2 - \tfrac{3}{5}\lambda + \tfrac{1}{5} = \frac{-\lambda^3}{5}\left(-\frac{1}{\lambda^3} + \frac{3}{\lambda^2} - \frac{7}{\lambda} + 5\right).$$

♡ 8.2.33. A square matrix A is called *nilpotent* if $A^k = \mathrm{O}$ for some $k \geq 1$. (a) Prove that the only eigenvalue of a nilpotent matrix is 0. (The converse is also true; see Exercise 8.6.13.) (b) Find examples where $A^{k-1} \neq \mathrm{O}$ but $A^k = \mathrm{O}$ for $k = 2, 3$, and in general.

Solution:

(a) If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{0} = A^k\mathbf{v} = \lambda^k\mathbf{v}$ and hence $\lambda^k = 0$.

(b) If A has size $(k+1)\times(k+1)$ and is all zero except for $a_{i,i+1} = 1$ on the supra-diagonal, i.e., a $k \times k$ Jordan block $J_{0,k}$ with zeros along the diagonal.

♡ 8.2.34. (a) Prove that every eigenvalue of a matrix A is also an eigenvalue of its transpose A^T. (b) Do they have the same eigenvectors? (c) Prove that if $\mathbf{v}$ is an eigenvector of A with eigenvalue λ and $\mathbf{w}$ is an eigenvector of A^T with a different eigenvalue $\mu \neq \lambda$, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors with respect to the dot product.
(d) Illustrate this result when (i) $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$, (ii) $A = \begin{pmatrix} 5 & -4 & 2 \\ 5 & -4 & 1 \\ -2 & 2 & -3 \end{pmatrix}$.

Solution:

(a) $\det(A^T - \lambda\,\mathrm{I}) = \det(A - \lambda\,\mathrm{I})^T = \det(A - \lambda\,\mathrm{I})$, and hence A and A^T have the same characteristic polynomial.

(b) No. See the examples.

(c) $\lambda\,\mathbf{v}\cdot\mathbf{w} = (A\mathbf{v})^T\mathbf{w} = \mathbf{v}^T A^T\mathbf{w} = \mu\,\mathbf{v}\cdot\mathbf{w}$, so if $\mu \neq \lambda$, $\mathbf{v}\cdot\mathbf{w} = 0$ and the vectors are orthogonal.

(d) (i) The eigenvalues are $1, 2$; the eigenvectors of A are $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$; the eigenvectors of A^T are $\mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $\mathbf{v}_1, \mathbf{w}_2$ are orthogonal, as are $\mathbf{v}_2, \mathbf{w}_1$. The eigenvalues are $1, -1, -2$; the eigenvectors of A are $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 1 \end{pmatrix}$; the eigenvectors of A^T are $\mathbf{w}_1 = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \mathbf{w}_2 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.
Note that $\mathbf{v}_i$ is orthogonal to $\mathbf{w}_j$ whenever $i \neq j$.

8.2.35. (a) Prove that every real 3×3 matrix has at least one real eigenvalue. (b) Find a real

4×4 matrix with no real eigenvalues. (c) Can you find a real 5×5 matrix with no real eigenvalues?

Solution: (a) The characteristic equation of a 3×3 matrix is a cubic polynomial, and hence has at least one real root. (b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ has eigenvalues $\pm\,\mathrm{i}$. (c) No, since the characteristic polynomial is degree 5 and hence has at least one real root.

8.2.36. (a) Show that if A is a matrix such that $A^4 = \mathrm{I}$, then the only possible eigenvalues of A are $1, -1$, i and $-\,\mathrm{i}$. (b) Give an example of a real matrix that has all four numbers as eigenvalues.

Solution:

(a) If $A\mathbf{v} = \lambda\mathbf{v}$, then $\mathbf{v} = A^4\mathbf{v} = \lambda^4\mathbf{v}$, and hence any eigenvalue must satisfy $\lambda^4 = 1$.

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

8.2.37. A *projection matrix* satisfies $P^2 = P$. Find all eigenvalues and eigenvectors of P.

Solution: If $P\mathbf{v} = \lambda\mathbf{v}$ then $P^2\mathbf{v} = \lambda^2\mathbf{v}$. Since $P\mathbf{v} = P^2\mathbf{v}$, we find $\lambda\mathbf{v} = \lambda^2\mathbf{v}$ and so, since $\mathbf{v} \neq \mathbf{0}$, $^2 = \lambda$ so the only eigenvalues are $\lambda = 0, 1$. All $\mathbf{v} \in \operatorname{rng} P$ are eigenvectors with eigenvalue 1 since if $\mathbf{v} = P\mathbf{u}$, then $P\mathbf{v} = P^2\mathbf{u} = P\mathbf{u} = \mathbf{v}$, whereas all $\mathbf{w} \in \ker P$ are null eigenvectors.

8.2.38. *True or false*: All $n \times n$ permutation matrices have real eigenvalues.

Solution: False. For example, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ has eigenvalues $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\,\mathrm{i}$.

8.2.39. (a) Show that if all the row sums of A are equal to 1, then A has 1 as an eigenvalue. (b) Suppose all the column sums of A are equal to 1. Does the same result hold? *Hint*: Use Exercise 8.2.34.

Solution:

(a) According to Exercise 1.2.29, if $\mathbf{z} = (1, 1, \ldots, 1)^T$, then $A\mathbf{z}$ is the vector of row sums of A, and hence, by the assumption, $A\mathbf{z} = \mathbf{z}$ so $\mathbf{z}$ is an eigenvector with eigenvalue 1.

(b) Yes, since the column sums of A are the row sums of A^T, and Exercise 8.2.34 says that A and A^T have the same eigenvalues.

8.2.40. Let Q be an orthogonal matrix. (a) Prove that if λ is an eigenvalue, then so is $1/\lambda$. (b) Prove that all its eigenvalues are complex numbers of modulus $|\,\lambda\,| = 1$. In particular, the only possible real eigenvalues of an orthogonal matrix are ± 1. (c) Suppose $\mathbf{v} = \mathbf{x} + \mathrm{i}\,\mathbf{y}$ is a complex eigenvector corresponding to a non-real eigenvalue. Prove that its real and imaginary parts are orthogonal vectors having the same Euclidean norm.

Solution:

(a) If $Q\mathbf{v} = \lambda\mathbf{v}$, then $Q^T\mathbf{v} = Q^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$ and so λ^{-1} is an eigenvalue of Q^T. Exercise 8.2.34 says that a matrix and its transpose have the same eigenvalues.

(b) If $Q\mathbf{v} = \lambda\mathbf{v}$, then, by Exercise 5.3.16, $\|\,\mathbf{v}\,\| = \|\,Q\mathbf{v}\,\| = |\,\lambda\,|\,\|\,\mathbf{v}\,\|$, and hence $|\,\lambda\,| = 1$. Note that this proof also applies to complex eigenvalues/eigenvectors using the Hermitian norm.

(c) If $e^{\mathrm{i}\,\theta} = \cos\theta + \mathrm{i}\,\sin\theta$ is the eigenvalue, then $Q\mathbf{x} = \cos\theta\mathbf{x} - \sin\theta\mathbf{y}$, $Q\mathbf{y} = \sin\theta\mathbf{x} + \cos\theta\mathbf{y}$, while $Q^{-1}\mathbf{x} = \cos\theta\mathbf{x} - \sin\theta\mathbf{y}$, $Q^{-1}\mathbf{y} = \sin\theta\mathbf{x} + \cos\theta\mathbf{y}$. Thus, $\cos^2\theta\|\mathbf{x}\|^2 + 2\cos\theta\sin\theta\mathbf{x}\cdot\mathbf{y} + \sin^2\theta\mathbf{y}^2 = \|Q^{-1}\mathbf{x}\|^2 = \|\mathbf{x}\|^2 = \|Q\mathbf{x}\|^2 = \cos^2\theta\|\mathbf{x}\|^2 - 2\cos\theta\sin\theta\mathbf{x}\cdot\mathbf{y} + \sin^2\theta\mathbf{y}^2$ and so for $\mathbf{x}\cdot\mathbf{y} = 0$ provided $\theta \neq 0, \pm\frac{1}{2}\pi, \pm\pi, \ldots$. Moreover, $\|\mathbf{x}\|^2 = \cos^2\theta\|\mathbf{x}\|^2 + \sin^2\theta\mathbf{y}^2$, while, by the same calculation, $\|\mathbf{y}\|^2 = \sin^2\theta\|\mathbf{x}\|^2 + \cos^2\theta\mathbf{y}^2$ which imply $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2$. For $\theta = \frac{1}{2}\pi$, we have $Q\mathbf{x} = -\mathbf{y}, Q\mathbf{y} = x$, and so $\mathbf{x}\cdot\mathbf{y} = -(Q\mathbf{y})\cdot(Q\mathbf{x}) = \mathbf{x}\cdot\mathbf{y}$ is also zero. ■

♢ 8.2.41. (*a*) Prove that every 3×3 proper orthogonal matrix has $+1$ as an eigenvalue. (*b*) *True or false*: An improper 3×3 orthogonal matrix has -1 as an eigenvalue.

Solution:

(*a*) According to Exercise 8.2.35, a 3×3 orthogonal matrix has at least one real eigenvalue, which by Exercise 8.2.40 must be ± 1. If the other two eigenvalues are complex conjugate, $\mu \pm \mathrm{i}\,\nu$, then the product of the eigenvalues is $\pm(\mu^2+\nu^2)$. Since this must equal the determinant of Q, which by assumption, is positive, we conclude that the real eigenvalue must be $+1$. Otherwise, all the eigenvalues of Q are real, and they cannot all equal -1 as otherwise its determinant would be negative.

(*b*) True. It must either have three real eigenvalues of ± 1, of which at least one must be -1 as otherwise its determinant would be $+1$, or a complex conjugate pair of eigenvalues $\lambda, \overline{\lambda}$, and its determinant is $-1 = \pm|\lambda|^2$, so its real eigenvalue must be -1 and its complex eigenvalues $\pm\mathrm{i}$.

♢ 8.2.42. (*a*) Show that the linear transformation defined by 3×3 proper orthogonal matrix corresponds to rotating through an angle around a line through the origin in $\mathbb{R}^3$ — the axis of the rotation. *Hint*: Use Exercise 8.2.41(*a*).

(*b*) Find the axis and angle of rotation of the orthogonal matrix $\begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ -\frac{4}{13} & \frac{12}{13} & \frac{3}{13} \\ -\frac{48}{65} & -\frac{5}{13} & \frac{36}{65} \end{pmatrix}$.

Solution: (a) The axis of the rotation is the eigenvector $\mathbf{v}$ corresponding to the eigenvalue $+1$. Since $Q\mathbf{v} = \mathbf{v}$, the rotation fixes the axis, and hence must rotate around it. Choosing an orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, where $\mathbf{u}_1$ is a unit eigenvector in the direction of the axis of rotation, while $\mathbf{u}_2 + \mathrm{i}\,\mathbf{u}_3$ is a complex eigenvector for the eigenvalue $e^{\mathrm{i}\,\theta}$. In this basis, Q has matrix form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$, where θ is the angle of rotation. (*b*) The axis is the eigenvector $\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$ for the eigenvalue 1. The complex eigenvalue is $\frac{7}{13} + \mathrm{i}\,\frac{2\sqrt{30}}{13}$, and so the angle is $\theta = \cos^{-1}\frac{7}{13} \approx 1.00219$.

8.2.43. Find all invariant subspaces, cf. Exercise 7.4.32, of a rotation in $\mathbb{R}^3$.

Solution: In general, besides the trivial invariant subspaces $\{\mathbf{0}\}$ and $\mathbb{R}^3$, the axis of rotation and its orthogonal complement plane are invariant. If the rotation is by 180°, then any line in the orthogonal complement plane is also invariant. If $R = \mathrm{I}$, then every subspace is invariant.

8.2.44. Suppose Q is an orthogonal matrix. (*a*) Prove that $K = 2\,\mathrm{I} - Q - Q^T$ is a positive semi-definite matrix. (*b*) Under what conditions is $K > 0$?

Solution:

(*a*) $(Q-\mathrm{I})^T(Q-\mathrm{I}) = Q^TQ - Q - Q^T + \mathrm{I} = 2\,\mathrm{I} - Q - Q^T = K$ and hence K is a Gram matrix, which is positive semi-definite by Theorem 3.28.

(*b*) The Gram matrix is positive definite if and only if $\ker(Q-\mathrm{I}) = \{\mathbf{0}\}$, which means that

Q does not have an eigenvalue of 1.

◇ 8.2.45. Prove that every *proper* affine isometry $F(\mathbf{x}) = Q\,\mathbf{x} + \mathbf{b}$ of $\mathbb{R}^3$, where $\det Q = +1$, is one of the following: (*a*) a *translation* $\mathbf{x} + \mathbf{b}$, (*b*) a *rotation* centered at some point of $\mathbb{R}^3$, or (*c*) a *screw* consisting of a rotation around an axis followed by a translation in the direction of the axis. *Hint*: Use Exercise 8.2.42.

Solution: If $Q = \mathrm{I}$, then we have a translation. Otherwise, we can write $F(\mathbf{x}) = Q(\mathbf{x} - \mathbf{c}) + \mathbf{c}$ in the form of a rotation around the center point $\mathbf{c}$ provided we can solve $(Q - \mathrm{I})\mathbf{c} = \mathbf{b}$. By the Fredholm alternative, this requires $\mathbf{b}$ to be orthogonal to $\operatorname{coker}(Q - \mathrm{I})$ which is also spanned by the rotation axis $\mathbf{v}$, i.e., the eigenvector for the eigenvalue $+1$ of $Q^T = Q^{-1}$. More generally, we write $F(\mathbf{x}) = Q(\mathbf{x} - \mathbf{c}) + \mathbf{c} + t\,\mathbf{v}$ and identify the affine map as a screw around the axis in the direction of $\mathbf{v}$ passing through $\mathbf{c}$.

♡ 8.2.46. Let M_n be the $n \times n$ tridiagonal matrix hose diagonal entries are all equal to 0 and whose sub- and super-diagonal entries all equal 1. (*a*) Find the eigenvalues and eigenvectors of M_2 and M_3 directly. (*b*) Prove that the eigenvalues and eigenvectors of M_n are explicitly given by

$$\lambda_k = 2\cos\frac{k\pi}{n+1}, \quad \mathbf{v}_k = \left(\sin\frac{k\pi}{n+1}, \quad \sin\frac{2k\pi}{n+1}, \quad \ldots \quad \sin\frac{nk\pi}{n+1}\right)^T, \quad k = 1, \ldots, n.$$

How do you know that there are no other eigenvalues?

Solution:

(*a*) For $M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: eigenvalues $1, -1$; eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. For $M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$: eigenvalues $-\sqrt{2}, 0, \sqrt{2}$; eigenvectors $\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

(*b*) The j^{th} entry of the eigenvalue equation $M_n\mathbf{v}_k = \lambda_k\mathbf{v}_k$ reads

$$\sin\frac{(j-1)k\pi}{n+1} + \sin\frac{(j+1)k\pi}{n+1} = 2\cos\frac{k\pi}{n+1}\,\sin\frac{jk\pi}{n+1},$$

which is a standard trigonometric identity: $\sin\alpha + \sin\beta = 2\cos\dfrac{\alpha-\beta}{2}\sin\dfrac{\alpha+\beta}{2}$. These are all the eigenvalues because an $n \times n$ matrix has at most n eigenvalues.

◇ 8.2.47. Let a, b be fixed scalars. Determine the eigenvalues and eigenvectors of the $n \times n$ tridiagonal matrix with all diagonal entries equal to a and all sub- and super-diagonal entries equal to b. *Hint*: See Exercises 8.2.18, 46.

Solution: We have $A = a\,\mathrm{I} + b\,M_n$, so by Exercises 8.2.18 and 8.2.46 it has the same eigenvectors as M_n, while its corresponding eigenvalues are $a + b\,\lambda_k = a + 2\,b\cos\dfrac{k\pi}{n+1}$ for $k = 1, \ldots, n$.

♡ 8.2.48. Find a formula for the eigenvalues of the tricirculant $n \times n$ matrix Z_n that has 1's on the sub- and super-diagonals as well as its $(1, n)$ and $(n, 1)$ entries, while all other entries are 0. *Hint*: Use Exercise 8.2.46 as a guide.

Solution:

$$\lambda_k = 2\cos\frac{2k\pi}{n}, \quad \mathbf{v}_k = \left(\cos\frac{2k\pi}{n}, \quad \cos\frac{4k\pi}{n}, \quad \cos\frac{6k\pi}{n}, \quad \ldots, \quad \cos\frac{2(n-1)k\pi}{n}, \quad 1\right)^T,$$

for $k = 1, \ldots, n$.

8.2.49. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$, and B an $m \times m$ matrix with eigenvalues $\mu_1, \ldots, \mu_l$. Show that the $(m+n)\times(m+n)$ block diagonal matrix $D = \begin{pmatrix} A & \mathrm{O} \\ \mathrm{O} & B \end{pmatrix}$ has eigenvalues $\lambda_1, \ldots, \lambda_k$, $\mu_1, \ldots, \mu_l$ and no others. How are the eigenvectors related?

Solution: Note first that if $A\mathbf{v} = \lambda\mathbf{v}$, then $D\begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} A\mathbf{v} \\ \mathbf{0} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}$, and so $\begin{pmatrix} \mathbf{v} \\ \mathbf{0} \end{pmatrix}$ is an eigenvector for D with eigenvalue λ. Similarly, each eigenvalue μ and eigenvector $\mathbf{w}$ of B gives an eigenvector $\begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}$ of D. Finally, to check that D has no other eigenvalue, we compute $D\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} A\mathbf{v} \\ B\mathbf{w} \end{pmatrix} = \lambda\begin{pmatrix} \mathbf{v} \\ \mathbf{w} \end{pmatrix}$ and hence, if $\mathbf{v} \neq \mathbf{0}$, then λ is an eigenvalue of A, while if $\mathbf{w} \neq \mathbf{0}$, then it must also be an eigenvalue for B.

♡ 8.2.50. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. (*a*) Prove that A satisfies its own characteristic equation, meaning $p_A(A) = A^2 - (\operatorname{tr} A)\, A + (\det A)\, \mathrm{I} = \mathrm{O}$. *Remark*: This result is a special case of the *Cayley–Hamilton Theorem*, to be developed in Exercise 8.6.16.
(*b*) Prove the inverse formula $A^{-1} = \dfrac{(\operatorname{tr} A)\, \mathrm{I} - A}{\det A}$ when $\det A \neq 0$. (*c*) Check the Cayley–Hamilton and inverse formulas when $A = \begin{pmatrix} 2 & 1 \\ -3 & 2 \end{pmatrix}$.

Solution:
(*a*) Follows by direct computation.
(*b*) Multiply the characteristic equation by A^{-1} and rearrange terms.
(*c*) $\operatorname{tr} A = 4, \det A = 7$ and one checks $A^2 - 4A + 7\,\mathrm{I} = \mathrm{O}$.

♡ 8.2.51. *Deflation*: Suppose A has eigenvalue λ and corresponding eigenvector $\mathbf{v}$. (*a*) Let $\mathbf{b}$ be any vector. Prove that the matrix $B = A - \mathbf{v}\,\mathbf{b}^T$ also has $\mathbf{v}$ as an eigenvector, now with eigenvalue $\lambda - \beta$ where $\beta = \mathbf{v} \cdot \mathbf{b}$. (*b*) Prove that if $\mu \neq \lambda - \beta$ is any other eigenvalue of A, then it is also an eigenvalue of B. *Hint*: Look for an eigenvector of the form $\mathbf{w} + c\,\mathbf{v}$, where $\mathbf{w}$ is the eigenvector of A. (*c*) Given a nonsingular matrix A with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $\lambda_1 \neq \lambda_j$, $j \geq 2$, explain how to construct a *deflated* matrix B whose eigenvalues are $0, \lambda_2, \ldots, \lambda_n$. (*d*) Try out your method on the matrices $\begin{pmatrix} 3 & 3 \\ 1 & 5 \end{pmatrix}$ and $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

Solution:
(*a*) $B\mathbf{v} = (A - \mathbf{v}\,\mathbf{b}^T)\mathbf{v} = A\mathbf{v} - (\mathbf{b} \cdot \mathbf{v})\mathbf{v} = (\lambda - \beta)\mathbf{v}$.
(*b*) $B(\mathbf{w} + c\mathbf{v}) = (A - \mathbf{v}\,\mathbf{b}^T)(\mathbf{w} + c\mathbf{v}) = \mu\mathbf{w} + (c(\lambda - \beta) - \mathbf{b} \cdot \mathbf{w})\mathbf{v} = \mu(\mathbf{w} + c\mathbf{v})$ provided $c = \mathbf{b} \cdot \mathbf{w}/(\lambda - \beta - \mu)$.
(*c*) Set $B = A - \lambda_1 \mathbf{v}_1 \mathbf{b}^T$ where $\mathbf{v}_1$ is the first eigenvector of A and $\mathbf{b}$ is any vector such that $\mathbf{b} \cdot \mathbf{v}_1 = 1$. For example, we can set $\mathbf{b} = \mathbf{v}_1/\|\,\mathbf{v}_1\,\|^2$. (Weilandt deflation, [**12**], chooses $\mathbf{b} = \mathbf{r}_j/(\lambda_1 v_{1,j})$ where $v_{1,j}$ is any nonzero entry of $\mathbf{v}_1$ and $\mathbf{r}_j$ is the corresponding row of A.)
(*d*) (*i*) The eigenvalues of A are $6, 2$ and the eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix}$. The deflated matrix $B = A - \dfrac{\lambda_1 \mathbf{v}_1 \mathbf{v}_1^T}{\|\,\mathbf{v}_1\,\|^2} = \begin{pmatrix} 0 & 0 \\ -2 & 2 \end{pmatrix}$ has eigenvalues $0, 2$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
(*ii*) The eigenvalues of A are $4, 3, 1$ and the eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. The de-

flated matrix $B = A - \dfrac{\lambda_1 \mathbf{v}_1 \mathbf{v}_1^T}{\| \mathbf{v}_1 \|^2} = \begin{pmatrix} \frac{5}{3} & \frac{1}{3} & -\frac{4}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{4}{3} & \frac{1}{3} & \frac{5}{3} \end{pmatrix}$ has eigenvalues $0, 3, 1$ and the eigenvectors $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

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diag

8.3. Eigenvector Bases and Diagonalization.

8.3.1. Which of the following are complete eigenvalues for the indicated matrix? What is the dimension of the associated eigenspace? (*a*) 3, $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, (*b*) 2, $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, (*c*) 1, $\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (*d*) 1, $\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, (*e*) -1, $\begin{pmatrix} -1 & -4 & -4 \\ 0 & -1 & 0 \\ 0 & 4 & 3 \end{pmatrix}$, (*f*) $-\mathrm{i}$, $\begin{pmatrix} -\mathrm{i} & 1 & 0 \\ -\mathrm{i} & 1 & -1 \\ 0 & 0 & -\mathrm{i} \end{pmatrix}$, (*g*) -2, $\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, (*h*) 1, $\begin{pmatrix} 1 & -1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$.

Solution:

(*a*) Complete. $\dim = 1$ with basis $(1, 1)^T$.
(*b*) Not complete. $\dim = 1$ with basis $(1, 0)^T$.
(*c*) Complete. $\dim = 1$ with basis $(0, 1, 0)^T$.
(*d*) Not an eigenvalue.
(*e*) Complete. $\dim = 2$ with basis $(1, 0, 0)^T, (0, -1, 1)^T$.
(*f*) Complete. $\dim = 1$ with basis $(\mathrm{i}, 0, 1)^T$.
(*g*) Not an eigenvalue.
(*h*) Not complete. $\dim = 1$ with basis $(1, 0, 0, 0, 0)^T$.

8.3.2. Find the eigenvalues and a basis for the each of the eigenspaces of the following matrices. Which are complete?
(*a*) $\begin{pmatrix} 4 & -4 \\ 1 & 0 \end{pmatrix}$, (*b*) $\begin{pmatrix} 6 & -8 \\ 4 & -6 \end{pmatrix}$, (*c*) $\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$, (*d*) $\begin{pmatrix} \mathrm{i} & -1 \\ 1 & \mathrm{i} \end{pmatrix}$, (*e*) $\begin{pmatrix} 4 & -1 & -1 \\ 0 & 3 & 0 \\ 1 & -1 & 2 \end{pmatrix}$, (*f*) $\begin{pmatrix} -6 & 0 & -8 \\ -4 & 2 & -4 \\ 4 & 0 & 6 \end{pmatrix}$, (*g*) $\begin{pmatrix} -2 & 1 & -1 \\ 5 & -3 & 6 \\ 5 & -1 & 4 \end{pmatrix}$, (*h*) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}$, (*i*) $\begin{pmatrix} -1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ -1 & -4 & 1 & -2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Solution:

(*a*) Eigenvalue: 2. Eigenvector: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Not complete.

(*b*) Eigenvalues: $2, -2$. Eigenvectors: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Complete.

(*c*) Eigenvalues: $1 \pm 2\,\mathrm{i}$. Eigenvectors: $\begin{pmatrix} 1 \pm \mathrm{i} \\ 2 \end{pmatrix}$. Complete.

(*d*) Eigenvalues: $0, 2\,\mathrm{i}$. Eigenvectors: $\begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}$. Complete.

(*e*) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Not complete.

(*f*) Eigenvalue 2 has eigenspace basis $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Eigenvalue -2 has $\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$. Complete.

(*g*) Eigenvalue 3 has eigenspace basis $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Eigenvalue -2 has $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Not complete.

(*h*) Eigenvalue 0 has $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Eigenvalue -1 has $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Eigenvalue 1 has $\begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. Complete.

(*i*) Eigenvalue 0 has eigenspace basis $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$. Eigenvalue 2 has $\begin{pmatrix} -1 \\ 1 \\ -5 \\ 1 \end{pmatrix}$. Not complete.

8.3.3. Which of the following matrices admit eigenvector bases of $\mathbb{R}^n$? For those that do, exhibit such a basis. If not, what is the dimension of the subspace of $\mathbb{R}^n$ spanned by the eigenvectors? (a) $\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, (*d*) $\begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 4 & -4 & -1 \end{pmatrix}$, (*e*) $\begin{pmatrix} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & -1 \end{pmatrix}$, (*f*) $\begin{pmatrix} 2 & 0 & 0 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$, (*g*) $\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, (*h*) $\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

Solution:

(a) Eigenvalues: $-2, 4$; the eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a basis for $\mathbb{R}^2$.

(*b*) Eigenvalues: $1 - 3\,\mathrm{i}$, $1 + 3\,\mathrm{i}$; the eigenvectors $\begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -\mathrm{i} \\ 1 \end{pmatrix}$, are not real, so the dimension is 0.

(*c*) Eigenvalue: 1; there is only one eigenvector $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ spanning a one-dimensional subspace of $\mathbb{R}^2$.

(*d*) The eigenvalue 1 has eigenvector $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, while the eigenvalue -1 has eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. The eigenvectors form a basis for $\mathbb{R}^3$.

(*e*) The eigenvalue 1 has eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, while the eigenvalue -1 has eigenvector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. The eigenvectors span a two-dimensional subspace of $\mathbb{R}^3$.

(f) $\lambda = -2, 0, 2$. The eigenvectors are $\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix}$, forming a basis for $\mathbb{R}^3$.

(g) the eigenvalues are $\mathrm{i}, -\mathrm{i}, 1$. The eigenvectors are $\begin{pmatrix} -\mathrm{i} \\ 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \mathrm{i} \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. The real eigenvectors span only a one-dimensional subspace of $\mathbb{R}^3$.

(h) The eigenvalues are $-1, 1, -\mathrm{i} -1, -\mathrm{i} +1$. The eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 4 \\ 3 \\ 2 \\ 6 \end{pmatrix}$, $\begin{pmatrix} -1 \\ \mathrm{i} \\ -\mathrm{i} \\ 1 \end{pmatrix}$, $\begin{pmatrix} -1 \\ -\mathrm{i} \\ \mathrm{i} \\ 1 \end{pmatrix}$. The real eigenvectors span a two-dimensional subspace of $\mathbb{R}^4$.

8.3.4. Answer Exercise 8.3.3 with $\mathbb{R}^n$ replaced by $\mathbb{C}^n$.

Solution: Cases (a,b,d,f,g,h) all have eigenvector bases of $\mathbb{C}^n$.

8.3.5. (a) Give an example of a 3×3 matrix that only has 1 as an eigenvalue, and has only one linearly independent eigenvector. (b) Give an example that has two linearly independent eigenvectors.

Solution: (a) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, ($b$) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

8.3.6. *True or false*: (a) Every diagonal matrix is complete. (b) Every upper triangular matrix is complete.

Solution: (a) True. The standard basis vectors are eigenvectors. (b) False. The Jordan matrix 1 101 is incomplete since $\mathbf{e}_1$ is the only eigenvector.

8.3.7. Prove that if A is a complete matrix, so is $c\,A + d\,\mathrm{I}$, where c, d are any scalars. *Hint*: Use Exercise 8.2.18.

Solution: According to Exercise 8.2.18, every eigenvector of A is an eigenvector of $c\,A + d\,\mathrm{I}$ with eigenvalue $c\lambda + d$, and hence if A has a basis of eigenvectors, so does $c\,A + d\,\mathrm{I}$.

8.3.8. (a) Prove that if A is complete, so is A^2. (b) Give an example of an incomplete matrix A such that A^2 is complete.

Solution: (a) Every eigenvector of A is an eigenvector of A^2 with eigenvalue λ^2, and hence if A has a basis of eigenvectors, so does A^2. (b) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ with $A^2 = \mathrm{O}$.

◇ 8.3.9. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_n$ forms an eigenvector basis for the complete matrix A, with $\lambda_1, \ldots, \lambda_n$ the corresponding eigenvalues. Prove that *every* eigenvalue of A is one of the $\lambda_1, \ldots, \lambda_n$.

Solution: Suppose $A\mathbf{v} = \lambda\mathbf{v}$. Write $\mathbf{v} = \sum_{i=1}^{n} c_i\,\mathbf{v}_i$. Then $A\mathbf{v} = \sum_{i=1}^{n} c_i\,\lambda_i\,\mathbf{v}_i$ and hence, by linear independence, $\lambda_i\,c_i = \lambda\,c_i$. THus, either $\lambda = \lambda_i$ or $c_i = 0$. *Q.E.D.*

8.3.10. (a) Prove that if λ is an eigenvalue of A, then λ^n is an eigenvalue of A^n. (b) State and prove a converse if A is complete. *Hint*: Use Exercise 8.3.9. (The completeness hypothesis is not essential, but this is harder, relying on the Jordan canonical form.)

Solution: (a) If $A\mathbf{v} = \lambda\mathbf{v}$, then, by induction, $A^n\mathbf{v} = \lambda^n\mathbf{v}$, and hence $\mathbf{v}$ is an eigenvector with eigenvalue λ^n. (b) Conversely, if A is complete and A^n has eigenvalue μ, then there is a (complex) n^{th} root $\lambda = \sqrt[n]{\mu}$ that is an eigenvalue of A. Indeed, the eigenvector basis of A is an eigenvector basis of A^n, and hence, using Exercise 8.3.9, every eigenvalue of A^n is the n^{th} power of an eigenvalue of A.

♢ 8.3.11. Show that if A is complete, then any similar matrix $B = S^{-1} A S$ is also complete.

Solution: As in Exercise 8.2.31, if $\mathbf{v}$ is an eigenvector of A then $S^{-1}\mathbf{v}$ is an eigenvector of B. Moreover, if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form a basis, so do $S^{-1}\mathbf{v}_1, \ldots, S^{-1}\mathbf{v}_n$; see Exercise 2.4.21 for details.

8.3.12. Let U be an upper triangular matrix with all its diagonal entries equal. Prove that U is complete if and only if U is a diagonal matrix.

Solution: According to Exercise 8.2.16, its only eigenvalue is λ, the common value of its diagonal entries, and so all eigenvectors belong to $\ker(U - \lambda\,\mathrm{I}\,)$. Thus U is complete if and only if $\dim\ker(U - \lambda\,\mathrm{I}\,) = n$ if and only if $U - \lambda\,\mathrm{I} = \mathrm{O}$. *Q.E.D.*

8.3.13. Show that each eigenspace of an $n \times n$ matrix A is an invariant subspace, as defined in Exercise 7.4.32.

Solution:
Let $V = \ker(A - \lambda\,\mathrm{I}\,)$. If $\mathbf{v} \in V$, then $A\mathbf{v} \in V$ since $(A - \lambda\,\mathrm{I}\,)A\mathbf{v} = A(A - \lambda\,\mathrm{I}\,)\mathbf{v} = \mathbf{0}$. *Q.E.D.*

♢ 8.3.14. (*a*) Prove that if $\mathbf{v} = \mathbf{x} \pm \mathrm{i}\,\mathbf{y}$ is a complex conjugate pair of eigenvectors of a real matrix A corresponding to complex conjugate eigenvalues $\mu \pm \mathrm{i}\,\nu$ with $\nu \neq 0$, then $\mathbf{x}$ and $\mathbf{y}$ are linearly independent real vectors. (*b*) More generally, if $\mathbf{v}_j = \mathbf{x}_j \pm \mathrm{i}\,\mathbf{y}_j$, $j = 1, \ldots, k$ are complex conjugate pairs of eigenvectors corresponding to *distinct* pairs of complex conjugate eigenvalues $\mu_j \pm \mathrm{i}\,\nu_j$, $\nu_j \neq 0$, then the real vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_k$ are linearly independent.

Solution:

(a) Let $\mu \pm \mathrm{i}\,\nu$ be the corresponding eigenvalues. Then the complex eigenvalue equation $A(\mathbf{x} + \mathrm{i}\,\mathbf{y}) = (\mu + \mathrm{i}\,\nu)(\mathbf{x} + \mathrm{i}\,\mathbf{y})$ implies that
$$A\mathbf{x} = \mu\mathbf{x} - \nu\mathbf{y} \qquad \text{and} \qquad A\mathbf{y} = \nu\mathbf{x} + \mu\mathbf{y}.$$
Now, suppose $c\mathbf{x} + d\mathbf{y} = \mathbf{0}$ for some $(c, d) \neq (0, 0)$. Then
$$\mathbf{0} = A(c\mathbf{x} + d\mathbf{y}) = (c\mu + d\nu)\mathbf{x} + (-c\nu + d\mu)\mathbf{y}$$
The determinant of the coefficient matrix of these two sets of equations for $\mathbf{x}, \mathbf{y}$ is $\det\begin{pmatrix} c & d \\ c\mu + d\nu & -c\nu + d\mu \end{pmatrix} = -(c^2 + d^2)\nu \neq 0$ because we are assuming the eigenvalues are truly complex. This implies $\mathbf{x} = \mathbf{y} = \mathbf{0}$, which contradicts the fact that we have an eigenvector.

(*b*) ■ This is proved by induction. Suppose we know $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_1, \ldots, \mathbf{y}_{k-1}$ are linearly independent. If $\mathbf{x}_1, \ldots, \mathbf{x}_k, \mathbf{y}_1, \ldots, \mathbf{y}_k$ were linearly dependent, there would exist $(c_k, d_k) \neq (0, 0)$ such that . $\mathbf{z}_k = c_k\mathbf{x}_k + d_k\mathbf{y}_k$ is some linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_1, \ldots, \mathbf{y}_{k-1}$. But then $A\mathbf{z}_k = (c_k\mu + d_k\nu)\mathbf{x}_k + (-c_k\nu + d_k\mu)\mathbf{y}_k$ is also a linear combination, as is $A^2\mathbf{z}_k = (c_k(\mu^2 - \nu^2) + 2d_k\mu\nu)\mathbf{x}_k + (-2c_k\mu\nu + d_k(\mu^2 - \nu^2))\mathbf{y}_k$. Since the coefficient matrix of the first two such vectors is nonsingular, a suitable linear combination would vanish: $\mathbf{0} = a\mathbf{z}_k + bA\mathbf{z}_k + cA^2\mathbf{z}_k$, which would give a vanishing linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}, \mathbf{y}_1, \ldots, \mathbf{y}_{k-1}$, which, by the induction hypothesis, must be trivial. A little more work demonstrates that this implies $\mathbf{z}_k = \mathbf{0}$, and so, in contradiction to part (*b*), would imply $\mathbf{x}_k, \mathbf{y}_k$ are linearly dependent. *Q.E.D.*

8.3.15. Diagonalize the following matrices: (a) $\begin{pmatrix} 3 & -9 \\ 2 & -6 \end{pmatrix}$, (*b*) $\begin{pmatrix} 5 & -4 \\ 2 & -1 \end{pmatrix}$, (*c*) $\begin{pmatrix} -4 & -2 \\ 5 & 2 \end{pmatrix}$,

$(d)\ \begin{pmatrix} -2 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix}$, $(e)\ \begin{pmatrix} 8 & 0 & -3 \\ -3 & 0 & -1 \\ 3 & 0 & -2 \end{pmatrix}$, $(f)\ \begin{pmatrix} 3 & 3 & 5 \\ 5 & 6 & 5 \\ -5 & -8 & -7 \end{pmatrix}$, $(g)\ \begin{pmatrix} 2 & 5 & 5 \\ 0 & 2 & 0 \\ 0 & -5 & -3 \end{pmatrix}$,

$(h)\ \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}$, $(i)\ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $(j)\ \begin{pmatrix} 2 & 1 & -1 & 0 \\ -3 & -2 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

Solution:

(a) $S = \begin{pmatrix} 3 & 3 \\ 1 & 2 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 \\ 0 & -3 \end{pmatrix}$.

(b) $S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(c) $S = \begin{pmatrix} -\frac{3}{5} + \frac{1}{5}\,\mathrm{i} & -\frac{3}{5} - \frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 + \mathrm{i} & 0 \\ 0 & -1 - \mathrm{i} \end{pmatrix}$.

(d) $S = \begin{pmatrix} 1 & 1 & -\frac{1}{10} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(e) $S = \begin{pmatrix} 0 & 21 & 1 \\ 1 & -10 & 6 \\ 0 & 7 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix}$.

(f) $S = \begin{pmatrix} -\frac{1}{5} - \frac{3}{5}\,\mathrm{i} & -\frac{1}{5} + \frac{3}{5}\,\mathrm{i} & -1 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 + 3\,\mathrm{i} & 0 & 0 \\ 0 & 2 - 3\,\mathrm{i} & 0 \\ 0 & 0 & -2 \end{pmatrix}$.

(g) $S = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

(h) $S = \begin{pmatrix} -4 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 6 & 0 & 0 \\ 12 & 0 & 0 & 0 \end{pmatrix}$, $D = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

(i) $S = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(j) $S = \begin{pmatrix} -1 & 1 & \frac{3}{2}\,\mathrm{i} & -\frac{3}{2}\,\mathrm{i} \\ 1 & -3 & -\frac{1}{2} - 2\,\mathrm{i} & -\frac{1}{2} + 2\,\mathrm{i} \\ 0 & 0 & 1 + \mathrm{i} & 1 - \mathrm{i} \\ 0 & 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \end{pmatrix}$.

8.3.16. Diagonalize the *Fibonacci matrix* $F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution: $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{pmatrix}$.

8.3.17. Diagonalize the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of rotation through 90°. How would you interpret the result?

Solution: $S = \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$. A rotation does not stretch any real vectors, but somehow corresponds to two complex stretches.

8.3.18. Diagonalize the rotation matrices (*a*) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} \frac{5}{13} & 0 & \frac{12}{13} \\ 0 & 1 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} \end{pmatrix}$.

Solution:

(*a*) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & -\mathrm{i} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & 0 & 0 \\ 0 & -\mathrm{i} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ \frac{\mathrm{i}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$,

(*b*) $\begin{pmatrix} \frac{5}{13} & 0 & \frac{12}{13} \\ 0 & 1 & 0 \\ -\frac{12}{13} & 0 & \frac{5}{13} \end{pmatrix} = \begin{pmatrix} -\mathrm{i} & \mathrm{i} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{5+12\,\mathrm{i}}{13} & 0 & 0 \\ 0 & \frac{5-12\,\mathrm{i}}{13} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ -\frac{\mathrm{i}}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}$.

8.3.19. Which of the following matrices have real diagonal forms? (*a*) $\begin{pmatrix} -2 & 1 \\ 4 & 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} 1 & 2 \\ -3 & 1 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & 3 & 2 \\ -1 & 1 & -1 \\ 1 & -3 & -1 \end{pmatrix}$, (*e*) $\begin{pmatrix} 3 & -8 & 2 \\ -1 & 2 & 2 \\ 1 & -4 & 2 \end{pmatrix}$, (*f*) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix}$.

Solution:
(*a*) Yes, distinct real eigenvalues $-3, 2$.
(*b*) No, complex eigenvalues $1 \pm \mathrm{i}\sqrt{6}$.
(*c*) No, complex eigenvalues $1, -\frac{1}{2} \pm \frac{\sqrt{5}}{2}\,\mathrm{i}$.
(*d*) No, incomplete eigenvalue 1 (and complete eigenvalue -2).
(*e*) Yes, distinct real eigenvalues $1, 2, 4$.
(*f*) Yes, complete real eigenvalues $1, -1$.

8.3.20. Diagonalize the following complex matrices:

(*a*) $\begin{pmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix}$, (*b*) $\begin{pmatrix} 1-\mathrm{i} & 0 \\ \mathrm{i} & 2+\mathrm{i} \end{pmatrix}$, (*c*) $\begin{pmatrix} 2-\mathrm{i} & 2+\mathrm{i} \\ 3-\mathrm{i} & 1+\mathrm{i} \end{pmatrix}$, (*d*) $\begin{pmatrix} -\mathrm{i} & 0 & 1 \\ -\mathrm{i} & 1 & -1 \\ 1 & 0 & -\mathrm{i} \end{pmatrix}$.

Solution:
(*a*) $S = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & -1+\mathrm{i} \end{pmatrix}$.
(*b*) $S = \begin{pmatrix} -2+\mathrm{i} & 0 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1-\mathrm{i} & 0 \\ 0 & 2+\mathrm{i} \end{pmatrix}$.
(*c*) $S = \begin{pmatrix} 1 & -\frac{1}{2} - \frac{1}{2}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix}$.
(*d*) $S = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1-\mathrm{i} & \frac{3}{5} + \frac{1}{5}\,\mathrm{i} \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\mathrm{i} & 0 \\ 0 & 0 & -1-\mathrm{i} \end{pmatrix}$.

8.3.21. Write down a real matrix that has (*a*) eigenvalues $-1, 3$ and corresponding eigenvectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, (*b*) eigenvalues $0, 2, -2$ and associated eigenvectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$; (*c*) an eigenvalue of 3 and corresponding eigenvectors $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$; (*d*) an eigenvalue

$-1+2\,\mathrm{i}$ and corresponding eigenvector $\begin{pmatrix} 1+\mathrm{i} \\ 3\,\mathrm{i} \end{pmatrix}$; (*e*) an eigenvalue -2 and corresponding eigenvector $\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$; (*f*) an eigenvalue $3+\mathrm{i}$ and corresponding eigenvector $\begin{pmatrix} 1 \\ 2\,\mathrm{i} \\ -1-\mathrm{i} \end{pmatrix}$.

Solution: Use $A = S\Lambda S^{-1}$. For parts (*e*), (*f*) you can choose any other eigenvalues and eigenvectors you want to fill in S and Λ. (*a*) $\begin{pmatrix} 7 & 4 \\ -8 & -5 \end{pmatrix}$, (*b*) $\begin{pmatrix} 6 & 6 & -2 \\ -2 & -2 & 0 \\ 6 & 6 & -4 \end{pmatrix}$, (*c*) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$,

(*d*) $\begin{pmatrix} 1 & -\frac{4}{3} \\ 6 & -3 \end{pmatrix}$, (*e*) example: $\begin{pmatrix} 0 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, (*f*) example: $\begin{pmatrix} 3 & \frac{1}{2} & 0 \\ -2 & 3 & 0 \\ -2 & -2 & 0 \end{pmatrix}$.

8.3.22. A matrix A has eigenvalues -1 and 2 and associated eigenvectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Write down the matrix form of the linear transformation $L[\mathbf{u}] = A\mathbf{u}$ in terms of (*a*) the standard basis $\mathbf{e}_1, \mathbf{e}_2$; (*b*) the basis consisting of its eigenvectors; (*c*) the basis $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Solution: (a) $\begin{pmatrix} 11 & -6 \\ 18 & -10 \end{pmatrix}$ (*b*) $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$, (*c*) $\begin{pmatrix} -4 & -6 \\ 3 & 5 \end{pmatrix}$.

♢ 8.3.23. Prove that two complete matrices A, B have the same eigenvalues (with multiplicities) if and only if they are similar, i.e., $B = S^{-1} A S$ for some nonsingular matrix S.

Solution: Let S_1 be the eigenvector matrix for A and S_2 the eigenvector matrix for B. Thus, by the hypothesis $S_1^{-1} A S_1 = \Lambda = S_2^{-1} B S_2$ and hence $A = S_1 S_2^{-1} B S_2 S_1^{-1} = S^{-1} B S$ where $S = S_2 S_1^{-1}$.

8.3.24. Let B be obtained from A by permuting both its rows and columns using the same permutation π, so $b_{ij} = a_{\pi(i),\pi(j)}$. Prove that A and B have the same eigenvalues. How are their eigenvectors related?

Solution: Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)^T$ be an eigenvector for A. Let $\widetilde{\mathbf{v}}$ be obtained by applying the permutation to the entries of $\mathbf{v}$, so $\widetilde{v}_i = v_{\pi(i)}$. Then the i^{th} entry of $B\widetilde{\mathbf{v}}$ is

$$\sum_{j=1}^{n} b_{ij}\,\widetilde{v}_j = \sum_{j=1}^{n} a_{\pi(i),\pi(j)}\, v_{\pi(j)} = \sum_{j=1}^{n} a_{\pi(i),j}\, v_j = \lambda\, v_{\pi(i)} = \lambda\,\widetilde{v}_i,$$

and hence $\widetilde{\mathbf{v}}$ is an eigenvector of B with eigenvalue λ. *Q.E.D.*

8.3.25. *True or false*: If A is a complete upper triangular matrix, then it has an upper triangular eigenvector matrix S.

Solution: True. Let $\lambda_j = a_{jj}$ denote the j^{th} diagonal entry of A, which is the same as the j^{th} eigenvalue. We will prove that the corresponding eigenvector is a linear combination of $\mathbf{e}_1, \ldots, \mathbf{e}_j$, which is equivalent to the eigenvector matrix S being upper triangular. We use induction on the size n. Since A is upper triangular, it leaves the subspace V spanned by $\mathbf{e}_1, \ldots, \mathbf{e}_{n-1}$ invariant, and hence its restriction is an $(n-1)\times(n-1)$ upper triangular matrix. Thus, by induction and completeness, A possesses $n-1$ eigenvectors of the required form. The remaining eigenvector $\mathbf{v}_n$ cannot belong to V (otherwise the eigenvectors would be linearly dependent) and hence must involve $\mathbf{e}_n$. *Q.E.D.*

8.3.26. Suppose the $n \times n$ matrix A is diagonalizable. How many different diagonal forms does it have?

Solution: The diagonal entries are all eigenvalues, and so are obtained from each other by permutation. If all eigenvalues are distinct, then there are $n!$ different diagonal forms — otherwise, if it has distinct eigenvalues of multiplicities $j_1, \ldots j_k$, there are $\dfrac{n!}{j_1! \,\cdots\, j_k!}$ distinct diagonal forms.

8.3.27. Characterize all complete matrices that are their own inverses: $A^{-1} = A$. Write down a non-diagonal example.

Solution: $A^2 = \mathrm{I}$ if and only if $D^2 = \mathrm{I}$, and so all its eigenvalues are ± 1. Examples: $A = \begin{pmatrix} 3 & -2 \\ 4 & -3 \end{pmatrix}$ with eigenvalues $1, -1$ and eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; or, even simpler, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

♡ 8.3.28. Two $n \times n$ matrices A, B are said to be *simultaneously diagonalizable* if there is a nonsingular matrix S such that both $S^{-1} A S$ and $S^{-1} B S$ are diagonal matrices. (*a*) Show that simultaneously diagonalizable matrices commute: $AB = BA$. (*b*) Prove that the converse is valid provided one of the matrices has no multiple eigenvalues. (*c*) Is every pair of commuting matrices simultaneously diagonalizable?

Solution: (*a*) If $A = S \Lambda S^{-1}$ and $B = S D S^{-1}$ where Λ, D are diagonal, then $AB = S \Lambda D S^{-1} = S D \Lambda S^{-1} = BA$, since diagonal matrices commute. (*b*) According to Exercise 1.2.12(*e*), the only matrices that commute with an $n \times n$ diagonal matrix with distinct entries is another diagonal matrix. Thus, if $AB = BA$, and $A = S \Lambda S^{-1}$ where all entries of Λ are distinct, then $D = S^{-1} B S$ commutes with Λ and hence is a diagonal matrix. (*c*) No, the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutes with the identity matrix, but is not diagonalizable. See also Exercise 1.2.14.

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evsym

8.4. Eigenvalues of Symmetric Matrices.

8.4.1. Find the eigenvalues and an orthonormal eigenvector basis for the following symmetric matrices:

(*a*) $\begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}$ (*b*) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, (*c*) $\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$, (*d*) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix}$, (*e*) $\begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}$.

Solution:

(*a*) eigenvalues: $5, -10$; eigenvectors: $\dfrac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \dfrac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

(*b*) eigenvalues: $7, 3$; eigenvectors: $\dfrac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(*c*) eigenvalues: $\dfrac{7 + \sqrt{13}}{2}, \dfrac{7 - \sqrt{13}}{2}$;

$$\text{eigenvectors: } \frac{2}{\sqrt{26 - 6\sqrt{13}}} \begin{pmatrix} \frac{3 - \sqrt{13}}{2} \\ 1 \end{pmatrix}, \frac{2}{\sqrt{26 + 6\sqrt{13}}} \begin{pmatrix} \frac{3 + \sqrt{13}}{2} \\ 1 \end{pmatrix}.$$

(d) eigenvalues: $6, 1, -4$; eigenvectors: $\begin{pmatrix} \frac{4}{5\sqrt{2}} \\ \frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{3}{5} \\ \frac{4}{5} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{4}{5\sqrt{2}} \\ -\frac{3}{5\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

(e) eigenvalues: $12, 9, 2$; eigenvectors: $\begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$.

8.4.2. Determine whether the following symmetric matrices are positive definite by computing their eigenvalues. Validate your conclusions by using the methods from Chapter 4.

(a) $\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 3 & 6 \end{pmatrix}$, ($c$) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, ($d$) $\begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix}$.

Solution:

(a) eigenvalues $\frac{5}{2} \pm \frac{1}{2}\sqrt{17}$; positive definite

(b) eigenvalues $-3, 7$; not positive definite

(c) eigenvalues $0, 1, 3$; positive semi-definite

(d) eigenvalues $6, 3 \pm \sqrt{3}$; positive definite.

8.4.3. Prove that a symmetric matrix is negative definite if and only if all its eigenvalues are negative.

Solution: Use the fact that $K = -N$ is positive definite and so has all positive eigenvalues. The eigenvalues of $N = -K$ are $-\lambda_j$ where λ_j are the eigenvalues of K. Alternatively, mimic the proof in the book for the positive definite case.

8.4.4. How many orthonormal eigenvector bases does a symmetric $n \times n$ matrix have?

Solution: If all eigenvalues are distinct, there are 2^n different bases, governed by the choice of sign in the unit eigenvectors $\pm \mathbf{u}_k$. If the eigenvalues are repeated, there are infinitely many, since any orthonormal basis of each eigenspace will contribute to an orthonormal eigenvector basis of the matrix.

8.4.5. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. ($a$) Write down necessary and sufficient conditions on the entries a, b, c, d that ensures that A has only real eigenvalues. (b) Verify that all symmetric 2×2 matrices satisfy your conditions.

Solution:

(a) The characteristic equation $p(\lambda) = \lambda^2 - (a+d)\lambda + (a\,d - b\,c) = 0$ has real roots if and only if its discriminant is non-negative: $0 \le (a+d)^2 - 4(a\,d - b\,c) = (a-d)^2 + 4\,b\,c$, which is the necessary and sufficient condition for real eigenvalues.

(b) If A is symmetric, then $b = c$ and so the discriminant is $(a-d)^2 + 4\,b^2 \ge 0$.

♡ 8.4.6. Let $A^T = -A$ be a real, skew-symmetric $n \times n$ matrix. (a) Prove that the only possible real eigenvalue of A is $\lambda = 0$. (b) More generally, prove that all eigenvalues λ of A are purely imaginary, i.e., Re $\lambda = 0$. (c) Explain why 0 is an eigenvalue of A whenever n is odd. (d) Explain why, if $n = 3$, the eigenvalues of $A \ne \mathrm{O}$ are $0, \mathrm{i}\,\omega, -\mathrm{i}\,\omega$ for some real ω. (e) Verify these facts for the particular matrices

(i) $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$, ($ii$) $\begin{pmatrix} 0 & 3 & 0 \\ -3 & 0 & -4 \\ 0 & 4 & 0 \end{pmatrix}$, ($iii$) $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$, ($iv$) $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$.

Solution:

(*a*) If $A\mathbf{v} = \lambda\mathbf{v}$ and $\mathbf{v} \neq \mathbf{0}$ is real, then
$$\lambda\|\,\mathbf{v}\,\|^2 = (A\mathbf{v})\cdot\mathbf{v} = (A\mathbf{v})^T\mathbf{v} = \mathbf{v}^T A^T\mathbf{v} = -\,\mathbf{v}^T A\mathbf{v} = -\,\mathbf{v}\cdot(A\mathbf{v}) = -\,\lambda\|\,\mathbf{v}\,\|^2,$$
and hence $\lambda = 0$.

(*b*) Using the Hermitian dot product,
$$\lambda\,\|\,\mathbf{v}\,\|^2 = (A\mathbf{v})\cdot\overline{\mathbf{v}} = \mathbf{v}^T A^T\overline{\mathbf{v}} = -\,\mathbf{v}^T A\,\overline{\mathbf{v}} = -\,\mathbf{v}\cdot(A\mathbf{v}) = -\,\overline{\lambda}\,\|\,\mathbf{v}\,\|^2,$$
and hence $\lambda = -\,\overline{\lambda}$, so λ is purely imaginary.

(*c*) Since $\det A = 0$, cf. Exercise 1.9.10, at least one of the eigenvalues of A must be 0.

(*d*) The characteristic polynomial of $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ is $-\lambda^3 + \lambda(a^2+b^2+c^2)$ and hence the eigenvalues are $0, \pm\,\mathrm{i}\,\sqrt{a^2+b^2+c^2}$, and so are all zero if and only if $A = \mathrm{O}$.

(*e*) The eigenvalues are: (*i*) $\pm 2\,\mathrm{i}$, (*ii*) $0, \pm 5\,\mathrm{i}$, (*iii*) $0, \pm\sqrt{3}\,\mathrm{i}$, (*iv*) $\pm 2\,\mathrm{i}, \pm 3\,\mathrm{i}$.

♡ 8.4.7. (*a*) Prove that every eigenvalue of a Hermitian matrix A, so $A^T = \overline{A}$ as in Exercise 3.6.49, is real. (*b*) Show that the eigenvectors corresponding to distinct eigenvalues are orthogonal under the Hermitian dot product on $\mathbb{C}^n$. (*c*) Find the eigenvalues and eigenvectors of the following Hermitian matrices, and verify orthogonality:
$$(i)\ \begin{pmatrix} 2 & \mathrm{i} \\ -\,\mathrm{i} & -2 \end{pmatrix}, \qquad (ii)\ \begin{pmatrix} 3 & 2-\mathrm{i} \\ 2+\mathrm{i} & -1 \end{pmatrix}, \qquad (iii)\ \begin{pmatrix} 0 & \mathrm{i} & 0 \\ -\,\mathrm{i} & 0 & \mathrm{i} \\ 0 & -\,\mathrm{i} & 0 \end{pmatrix}.$$

Solution:

(*a*) Let $A\mathbf{v} = \lambda\mathbf{v}$. Using the Hermitian dot product,
$$\lambda\|\,\mathbf{v}\,\|^2 = (A\mathbf{v})\cdot\overline{\mathbf{v}} = \mathbf{v}^T A^T\overline{\mathbf{v}} = \mathbf{v}^T\overline{A}\,\overline{\mathbf{v}} = \mathbf{v}\cdot(A\mathbf{v}) = \overline{\lambda}\|\,\mathbf{v}\,\|^2,$$
and hence $\lambda = \overline{\lambda}$, which implies that the eigenvalue λ is real.

(*b*) Let $A\mathbf{v} = \lambda\mathbf{v}$, $A\mathbf{w} = \mu\mathbf{w}$. Then
$$\lambda\mathbf{v}\cdot\mathbf{w} = (A\mathbf{v})\cdot\overline{\mathbf{w}} = \mathbf{v}^T A^T\overline{\mathbf{w}} = \mathbf{v}^T\overline{A}\,\overline{\mathbf{w}} = \mathbf{v}\cdot(A\mathbf{w}) = \mu\mathbf{v}\cdot\mathbf{w},$$
since μ is real. Thus, if $\lambda \neq \mu$ then $\mathbf{v}\cdot\mathbf{w} = 0$.

(*c*) (*i*) eigenvalues $\pm\sqrt{5}$; eigenvectors: $\begin{pmatrix} (2-\sqrt{5})\,\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} (2+\sqrt{5})\,\mathrm{i} \\ 1 \end{pmatrix}$.

(*ii*) eigenvalues $4, -2$; eigenvectors: $\begin{pmatrix} 2-\mathrm{i} \\ 1 \end{pmatrix}, \begin{pmatrix} -2+\mathrm{i} \\ 5 \end{pmatrix}$.

(*iii*) eigenvalues $0, \pm\sqrt{2}$; eigenvectors: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ \mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -\,\mathrm{i}\sqrt{2} \\ 1 \end{pmatrix}$.

♡ 8.4.8. Let $M > 0$ be a fixed positive definite $n \times n$ matrix. A nonzero vector $\mathbf{v} \neq \mathbf{0}$ is called a *generalized eigenvector* of the $n \times n$ matrix K if
$$K\mathbf{v} = \lambda M\mathbf{v}, \qquad \mathbf{v} \neq \mathbf{0}, \tag{8.31}$$
where the scalar λ is the corresponding *generalized eigenvalue*. (*a*) Prove that λ is a generalized eigenvalue of the matrix K if and only if it is an ordinary eigenvalue of the matrix $M^{-1}K$. How are the eigenvectors related? (*b*) Now suppose K is a symmetric matrix. Prove that its generalized eigenvalues are all real. *Hint*: First explain why this does *not* follow from part (a). Instead mimic the proof of part (*a*) of Theorem 8.20, using the weighted Hermitian inner product $\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle = \mathbf{v}^T M\,\overline{\mathbf{w}}$ in place of the dot product. (*c*) Show that if $K > 0$, then its generalized eigenvalues are all positive: $\lambda > 0$. (*d*) Prove that the eigenvectors corresponding to different generalized eigenvalues are orthogonal under the weighted inner product $\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle = \mathbf{v}^T M\,\mathbf{w}$. (*e*) Show that, if the matrix pair K, M has n distinct generalized eigenvalues, then the eigenvectors form an orthogonal basis for $\mathbb{R}^n$. *Re-*

mark: One can, by mimicking the proof of part (*c*) of Theorem 8.20, show that this holds even when there are repeated generalized eigenvalues.

Solution: (*a*) Rewrite (8.31) as $M^{-1}K\mathbf{v} = \lambda\mathbf{v}$, and so $\mathbf{v}$ is an eigenvector for $M^{-1}K$ with eigenvalue λ. (*b*) If $\mathbf{v}$ is an generalized eigenvector, then since K, M are real matrices, $K\overline{\mathbf{v}} = \overline{\lambda}\, M\,\overline{\mathbf{v}}$. Therefore,

$$\lambda \,\|\,\mathbf{v}\,\|^2 = \lambda\mathbf{v}^T M\,\overline{\mathbf{v}} = (\lambda M\mathbf{v})^T\,\overline{\mathbf{v}} = (K\mathbf{v})^T\,\overline{\mathbf{v}} = \mathbf{v}^T(K\overline{\mathbf{v}}) = \overline{\lambda}\mathbf{v}^T M\,\overline{\mathbf{v}} = \overline{\lambda}\,\|\,\mathbf{v}\,\|^2,$$

and hence λ is real. (*c*) If $K\mathbf{v} = \lambda M\mathbf{v}, K\mathbf{w} = \mu M\mathbf{w}$, with λ, μ and $\mathbf{v}, \mathbf{w}$ real, then

$$\lambda\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle = (\lambda M\mathbf{v})^T\,\mathbf{w} = (K\mathbf{v})^T\,\mathbf{w} = \mathbf{v}^T(K\mathbf{w}) = \mu\mathbf{v}^T M\mathbf{w} = \mu\,\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle,$$

and so if $\lambda \neq \mu$ then $\langle\,\mathbf{v}\,,\mathbf{w}\,\rangle = 0$, proving orthogonality. (*d*) If $K > 0$, then $\lambda\langle\,\mathbf{v}\,,\mathbf{v}\,\rangle = \mathbf{v}^T(\lambda M\,\mathbf{v}) = \mathbf{v}^T K\mathbf{v} > 0$, and so, by positive definiteness of M, $\lambda > 0$. (*e*) Part (*b*) proves that the eigenvectors are orthogonal with respect to the inner product induced by M, and so the result follows immediately from Theorem 5.5.

8.4.9. Compute the generalized eigenvalues and eigenvectors, as in (8.31), for the following matrix pairs. Verify orthogonality of the eigenvectors under the appropriate inner product.

(*a*) $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, (*b*) $K = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$,

(*c*) $K = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$, $M = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$, (*d*) $K = \begin{pmatrix} 6 & -8 & 3 \\ -8 & 24 & -6 \\ 3 & -6 & 99 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$,

(*e*) $K = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 1 \end{pmatrix}$, $M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, (*f*) $K = \begin{pmatrix} 5 & 3 & -5 \\ 3 & 3 & -1 \\ -5 & -1 & 9 \end{pmatrix}$, $M = \begin{pmatrix} 3 & 2 & -3 \\ 2 & 2 & -1 \\ -3 & -1 & 5 \end{pmatrix}$.

Solution:

(*a*) eigenvalues: $\frac{5}{3}, \frac{1}{2}$; eigenvectors: $\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$;

(*b*) eigenvalues: $2, \frac{1}{2}$; eigenvectors: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ 1 \end{pmatrix}$;

(*c*) eigenvalues: $7, 1$; eigenvectors: $\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$;

(*d*) eigenvalues: $12, 9, 2$; eigenvectors: $\begin{pmatrix} 6 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$;

(*e*) eigenvalues: $3, 1, 0$; eigenvectors: $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{pmatrix}$;

(*f*) 2 is a double eigenvalue with eigenvector basis $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, while 1 is a simple eigenvalue with eigenvector $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. For orthogonality you need to select an M orthogonal basis of the two-dimensional eigenspace, say by using Gram–Schmidt.

♢ 8.4.10. Let $L = L^*: \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint linear transformation with respect to the inner product $\langle\,\cdot\,,\cdot\,\rangle$. Prove that all its eigenvalues are real and the eigenvectors are orthogonal. *Hint*: Mimic the proof of Theorem 8.20 replacing the dot product by the given inner product.

Solution: If $L[\mathbf{v}] = \lambda \mathbf{v}$, then, using the inner product,

$$\lambda \| \mathbf{v} \|^2 = \langle L[\mathbf{v}], \mathbf{v} \rangle = \langle \mathbf{v}, L[\mathbf{v}] \rangle = \overline{\lambda} \| \mathbf{v} \|^2,$$

which proves that the eigenvalue λ is real. Similarly, if $L[w] = \mu : \mathbf{w}$, then

$$\lambda \langle \mathbf{v}, \mathbf{w} \rangle = \langle L[\mathbf{v}], \mathbf{w} \rangle = \langle \mathbf{v}, L[\mathbf{w}] \rangle = \mu \langle \mathbf{v}, \mathbf{w} \rangle,$$

and so if $\lambda \neq \mu$, then $\langle \mathbf{v}, \mathbf{w} \rangle = 0$.

♡ 8.4.11. The *difference map* $\Delta: \mathbb{C}^n \to \mathbb{C}^n$ is defined as $\Delta = S - \mathrm{I}$, where S is the shift map of Exercise 8.2.12. (*a*) Write down the matrix corresponding to Δ. (*b*) Prove that the sampled exponential vectors $\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_{n-1}$ from (5.84) form an eigenvector basis of Δ. What are the eigenvalues? (*c*) Prove that $K = \Delta^T \Delta$ has the same eigenvectors as Δ. What are its eigenvalues? (*d*) Is K positive definite? (*e*) According to Theorem 8.20 the eigenvectors of a symmetric matrix are real and orthogonal. Use this to explain the orthogonality of the sampled exponential vectors. But, why aren't they real?

Solution:

(*a*) $$\begin{pmatrix} -1 & 1 & 0 & & & & 0 & 0 \\ 0 & -1 & 1 & 0 & & & & 0 \\ & 0 & -1 & 1 & 0 & & & \\ & & \ddots & \ddots & \ddots & \ddots & & \\ & & & 0 & -1 & 1 & 0 & \\ 0 & & & & 0 & -1 & 1 & \\ 1 & 0 & & & & 0 & -1 & \end{pmatrix}.$$

(*b*) Using Exercise 8.2.12(*c*),

$$\Delta \boldsymbol{\omega}_k = (S - \mathrm{I})\, \boldsymbol{\omega}_k = \left(1 - e^{2 k \pi \mathrm{i}/n}\right) \boldsymbol{\omega}_k,$$

and so $\boldsymbol{\omega}_k$ is an eigenvector of Δ with corresponding eigenvalue $1 - e^{2 k \pi \mathrm{i}/n}$.

(*c*) Since S is an orthogonal matrix, $S^T = S^{-1}$, and so $S^T \boldsymbol{\omega}_k = e^{-2 k \pi \mathrm{i}/n} \boldsymbol{\omega}_k$. Therefore,

$$\begin{aligned} K \boldsymbol{\omega}_k &= (S^T - \mathrm{I})(S - \mathrm{I}) \boldsymbol{\omega}_k = (2\,\mathrm{I} - S - S^T) \boldsymbol{\omega}_k \\ &= \left(2 - e^{-2 k \pi \mathrm{i}/n} - e^{-2 k \pi \mathrm{i}/n}\right) \boldsymbol{\omega}_k = \left(2 - 2 \cos \frac{k \pi \mathrm{i}}{n}\right) \boldsymbol{\omega}_k, \end{aligned}$$

and hence $\boldsymbol{\omega}_k$ is an eigenvector of K with corresponding eigenvalue $2 - 2 \cos \dfrac{k \pi \mathrm{i}}{n}$.

(*d*) Yes, $K > 0$ since its eigenvalues are all positive; or note that $K = \Delta^T \Delta$ is a Gram matrix, with $\ker \Delta = \{\mathbf{0}\}$.

(*e*) Each eigenvalue $2 - 2 \cos \dfrac{k \pi \mathrm{i}}{n} = 2 - 2 \cos \dfrac{(n-k)\pi \mathrm{i}}{n}$ for $k \neq \frac{1}{2} n$ is double, with a two-dimensional eigenspace spanned by $\boldsymbol{\omega}_k$ and $\boldsymbol{\omega}_{n-k} = \overline{\boldsymbol{\omega}_k}$. The corrępsonding real eigenvectors are $\operatorname{Re} \boldsymbol{\omega}_k = \frac{1}{2} \boldsymbol{\omega}_k + \frac{1}{2} \boldsymbol{\omega}_{n-k}$ and $\operatorname{Im} \boldsymbol{\omega}_k = \frac{1}{2\mathrm{i}} \boldsymbol{\omega}_k - \frac{1}{2\mathrm{i}} \boldsymbol{\omega}_{n-k}$. On the other hand, if $k = \frac{1}{2} n$ (which requires that n be even), the eigenvector $\boldsymbol{\omega}_{n/2} = (1, -1, 1, -1, \ldots)^T$ is real.

♡ 8.4.12. An $n \times n$ *circulant matrix* has the form $C = \begin{pmatrix} c_0 & c_1 & c_2 & c_3 & \ldots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & c_2 & \ldots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & c_1 & \ldots & c_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ c_1 & c_2 & c_3 & c_4 & \ldots & c_0 \end{pmatrix}$,

in which the entries of each succeeding row are obtained by moving all the previous row's entries one slot to the right, the last entry moving to the front. (*a*) Check that the shift matrix S of Exercise 8.2.12, the difference matrix Δ, and its symmetric product K of Exercise 8.4.11 are all circulant matrices. (*b*) Prove that the sampled exponential vectors

$\boldsymbol{\omega}_0, \ldots, \boldsymbol{\omega}_{n-1}$, cf. (5.84) are eigenvectors of C. Thus, all circulant matrices have *the same eigenvectors*! What are the eigenvalues? (*c*) Prove that $\Omega_n^{-1} C\, \Omega_n = \Lambda$ where Ω_n is the Fourier matrix in Exercise 7.1.62 and Λ is the diagonal matrix with the eigenvalues of C along the diagonal. (*d*) Find the eigenvalues and eigenvectors of the following circulant matrices:

$$(i)\ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad (ii)\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}, \quad (iii)\ \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix}, \quad (iv)\ \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}.$$

(*e*) Find the eigenvalues of the tricirculant matrices in Exercise 1.7.13. Can you find a general formula for the $n \times n$ version? Explain why the eigenvalues must be real and positive. Does your formula reflect this fact? (*f*) Which of the preceding matrices are invertible? Write down a general criterion for checking the invertibility of circulant matrices.

Solution:

(*a*) The eigenvector equation

$$C \boldsymbol{\omega}_k = (c_0 + c_1 e^{2k\pi\,\mathrm{i}/n} + c_2 e^{4k\pi\,\mathrm{i}/n} + \cdots + + c_{n-1} e^{2(n-1)k\pi\,\mathrm{i}/n})\, \boldsymbol{\omega}_k$$

can either be proved directly, or by noting that

$$C = c_0\, \mathrm{I} + c_1 S + c_2 S^2 + \ \cdots\ + c_{n-1} S^{n-1},$$

and using Exercise 8.2.12(*c*).

(*b*) (*i*) Eigenvalues $3, -1$; eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

(*ii*) Eigenvalues $6, -\frac{3}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i}\,, , -\frac{3}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i}\,$; eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\,\mathrm{i} \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\,\mathrm{i} \end{pmatrix}$.

(*iii*) Eigenvalues $0, 2 - 2\,\mathrm{i}\,, 0, 2 + 2\,\mathrm{i}\,$; eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathrm{i} \\ -1 \\ \mathrm{i} \end{pmatrix}$.

(*iv*) Eigenvalues $0, 2, 4, 2$; eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ \mathrm{i} \\ -1 \\ -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -\mathrm{i} \\ -1 \\ \mathrm{i} \end{pmatrix}$.

(*c*) The eigenvalues are (*i*) $6, 3, 3$; (*ii*) $6, 4, 4, 2$; (*iii*) $6, \frac{7+\sqrt{5}}{2}, \frac{7+\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}, \frac{7-\sqrt{5}}{2}$; (*iv*) in the $n \times n$ case, they are $4 + 2\cos\dfrac{2k\pi}{n}$ for $k = 0, \ldots, n-1$. The eigenvalues are real and positive because the matrices are symmetric, positive definite.

(*d*) Cases (*i,ii*) in (*d*) and all matrices in part (*e*) are invertible. In general, an $n \times n$ circulant matrix is invertible if and only if none of the roots of the polynomial $c_0 + c_1 x + \cdots + c_{n-1} x^{n-1} = 0$ is an n^{th} root of unity: $x \neq e^{2k\pi/n}$.

8.4.13. Write out the spectral decomposition of the following matrices:

$$(a)\ \begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}, \quad (c)\ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad (d)\ \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Solution:

(a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \\ \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{pmatrix}.$

(b) $\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt2}{\sqrt{4-2\sqrt2}} & \frac{1+\sqrt2}{\sqrt{4+2\sqrt2}} \\ \frac{1}{\sqrt{4-2\sqrt2}} & \frac{1}{\sqrt{4+2\sqrt2}} \end{pmatrix} \begin{pmatrix} 3+\sqrt2 & 0 \\ 0 & 3-\sqrt2 \end{pmatrix} \begin{pmatrix} \frac{1-\sqrt2}{\sqrt{4-2\sqrt2}} & \frac{1}{\sqrt{4-2\sqrt2}} \\ \frac{1+\sqrt2}{\sqrt{4+2\sqrt2}} & \frac{1}{\sqrt{4+2\sqrt2}} \end{pmatrix}.$

(c) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt6} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{2}{\sqrt6} & 0 & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & \frac{2}{\sqrt6} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix}.$

(d) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt6} & 0 & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & \frac{1}{\sqrt2} & \frac{1}{\sqrt3} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{\sqrt6} & \frac{1}{\sqrt6} & \frac{1}{\sqrt6} \\ 0 & -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \end{pmatrix}.$

8.4.14. Write out the spectral factorization of the matrices listed in Exercise 8.4.1.

Solution:

(a) $\begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ \frac{1}{\sqrt5} & -\frac{2}{\sqrt5} \end{pmatrix}.$

(b) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix}.$

(c) $\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{13}}{\sqrt{26-6\sqrt{13}}} & \frac{3+\sqrt{13}}{\sqrt{26+6\sqrt{13}}} \\ \frac{2}{\sqrt{26-6\sqrt{13}}} & \frac{2}{\sqrt{26+6\sqrt{13}}} \end{pmatrix} \begin{pmatrix} \frac{7+\sqrt{13}}{2} & 0 \\ 0 & \frac{7-\sqrt{13}}{2} \end{pmatrix} \begin{pmatrix} \frac{3-\sqrt{13}}{\sqrt{26-6\sqrt{13}}} & \frac{2}{\sqrt{26-6\sqrt{13}}} \\ \frac{3+\sqrt{13}}{\sqrt{26+6\sqrt{13}}} & \frac{2}{\sqrt{26+6\sqrt{13}}} \end{pmatrix},$

(d) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5\sqrt2} & -\frac{3}{5} & -\frac{4}{5\sqrt2} \\ \frac{3}{5\sqrt2} & \frac{4}{5} & \frac{3}{5\sqrt2} \\ \frac{1}{\sqrt2} & 0 & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} \frac{4}{5\sqrt2} & \frac{3}{5\sqrt2} & \frac{1}{\sqrt2} \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5\sqrt2} & -\frac{3}{5\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix},$

(e) $\begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt6} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt2} \\ -\frac{1}{\sqrt6} & \frac{1}{\sqrt3} & \frac{1}{\sqrt2} \\ \frac{2}{\sqrt6} & \frac{1}{\sqrt3} & 0 \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} \\ -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \end{pmatrix}.$

8.4.15. Construct a symmetric matrix with the following eigenvectors and eigenvalues, or explain why none exists: (a) $\lambda_1 = 1$, $\mathbf{v}_1 = \left(\frac35, \frac45\right)^T$, $\lambda_2 = 3$, $\mathbf{v}_2 = \left(-\frac45, \frac35\right)^T$, (b) $\lambda_1 = -2$, $\mathbf{v}_1 = (1,-1)^T$, $\lambda_2 = 1$, $\mathbf{v}_2 = (1,1)^T$, (c) $\lambda_1 = 3$, $\mathbf{v}_1 = (2,-1)^T$, $\lambda_2 = -1$, $\mathbf{v}_2 = (-1,2)^T$, (d) $\lambda_1 = 2$, $\mathbf{v}_1 = (2,1)^T$, $\lambda_2 = 2$, $\mathbf{v}_2 = (1,2)^T$.

Solution: (a) $\begin{pmatrix} \frac{57}{25} & -\frac{24}{25} \\ -\frac{24}{25} & \frac{43}{25} \end{pmatrix}$; (b) $\begin{pmatrix} -\frac12 & \frac32 \\ \frac32 & -\frac12 \end{pmatrix}$; (c) none, since eigenvectors are not orthogonal; (d) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Even thought the given eigenvectors are not orthogonal, one can construct an orthogonal basis of the eigenspace.

8.4.16. Use the spectral factorization to diagonalize the following quadratic forms:
(a) $x^2 - 3xy + 5y^2$, (b) $3x^2 + 4xy + 6y^2$, (c) $x^2 + 8xz + y^2 + 6yz + z^2$,
(d) $\frac{3}{2}x^2 - xy - xz + y^2 + z^2$, (e) $6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2$.

Solution:

(a) $\frac{1}{2}\left(\frac{3}{\sqrt{10}}x + \frac{1}{\sqrt{10}}y\right)^2 + \frac{11}{2}\left(-\frac{1}{\sqrt{10}}x + \frac{3}{\sqrt{10}}y\right)^2 = \frac{1}{20}(3x+y)^2 + \frac{11}{20}(-x+3y)^2$,

(b) $7\left(\frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)^2 + \frac{11}{2}\left(-\frac{2}{\sqrt{5}}x + \frac{1}{\sqrt{5}}y\right)^2 = \frac{7}{5}(x+2y)^2 + \frac{2}{5}(-2x+y)^2$,

(c) $-4\left(\frac{4}{5\sqrt{2}}x + \frac{3}{5\sqrt{2}}y - \frac{1}{\sqrt{2}}z\right)^2 + \left(-\frac{3}{5}x + \frac{4}{5}y\right)^2 + 6\left(\frac{4}{5\sqrt{2}}x + \frac{3}{5\sqrt{2}}y + \frac{1}{\sqrt{2}}z\right)^2$
$= -\frac{2}{25}(4x+3y-5z)^2 + \frac{1}{25}(-3x+4y)^2 + \frac{3}{25}(4x+3y+5z)^2$,

(d) $\frac{1}{2}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right)^2 + \left(-\frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}}z\right)^2 + 2\left(-\frac{2}{\sqrt{6}}x + \frac{1}{\sqrt{6}}y + \frac{1}{\sqrt{6}}z\right)^2$
$= \frac{1}{6}(x+y+z)^2 + \frac{1}{2}(-y+z)^2 + \frac{1}{3}(-2x+y+z)^2$,

(e) $2\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 + 9\left(-\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right)^2 + 12\left(\frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{6}}y + \frac{2}{\sqrt{6}}z\right)^2$
$= (x+y)^2 + 3(-x+y+z)^2 + 2(x-y+2z)^2$.

♡ 8.4.17. (a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$. (b) Use the eigenvalues to compute the determinant of A. (c) Is A positive definite? Why or why not? (d) Find an orthonormal eigenvector basis of $\mathbb{R}^3$ determined by A or explain why none exists. (e) Write out the spectral factorization of A if possible. (f) Use orthogonality to write the vector $(1,0,0)^T$ as a linear combination of eigenvectors of A.

Solution:

(a) $\lambda_1 = \lambda_2 = 3, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \lambda_3 = 0, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$;

(b) $\det A = \lambda_1 \lambda_2 \lambda_3 = 0$.

(c) Only positive semi-definite; not positive definite since it has a zero eigenvalue.

(d) $\mathbf{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$;

(e) $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$;

(f) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{u}_1 - \frac{1}{\sqrt{6}}\mathbf{u}_2 + \frac{1}{\sqrt{3}}\mathbf{u}_3$.

8.4.18. *True or false*: A matrix with a real orthonormal eigenvector basis is symmetric.

Solution: True. If Q is the orthogonal matrix formed by the eigenvector basis, then $AQ = Q\Lambda$ where Λ is the diagonal eigenvalue matrix. Thus, $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$, which is symmetric.

◇ 8.4.19. Let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal basis of $\mathbb{R}^n$. Prove that it forms an eigenvector basis for some symmetric $n \times n$ matrix A. Can you characterize all such matrices?

Solution: The simplest is $A = \mathrm{I}$. More generally, any matrix of the form $A = S^T \Lambda S$, where $S = (\,\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_n\,)$ and Λ is any real diagonal matrix.

8.4.20. Prove that any quadratic form can be written as $\mathbf{x}^T A\,\mathbf{x} = \|\,\mathbf{x}\,\|^2 \left(\sum_{i=1}^{n} \lambda_i \cos^2 \theta_i \right)$, where λ_i are the eigenvalues of A and $\theta_i = \measuredangle\,(\mathbf{x}, \mathbf{v}_i)$ denotes the angle between $\mathbf{x}$ and the i^{th} eigenvector.

Solution: Using the spectral decomposition, we have $\mathbf{x}^T A\,\mathbf{x} = (Q^T\mathbf{x})^T \Lambda (Q^T\mathbf{x}) = \sum_{i=1}^{n} \lambda_i\, y_i^2$, where $y_i = \mathbf{u}_i \cdot \mathbf{x} = \|\,\mathbf{x}\,\|\, \cos\theta_i$ denotes the i^{th} entry of $Q^T\mathbf{x}$. *Q.E.D.*

8.4.21. An elastic body has stress tensor $T = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$. Find the principal stretches and principal directions of stretch.

Solution: Principal stretches = eigenvalues: $4 + \sqrt{3}, 4 - \sqrt{3}, 1$;

directions = eigenvectors $\left(1, -1+\sqrt{3}, 1\right)^T, \left(1, -1-\sqrt{3}, 1\right)^T, (\,-1, 0, 1\,)^T$.

◇ 8.4.22. Given a solid body spinning around its center of mass, the eigenvectors of its positive definite *inertia tensor* prescribe three mutually orthogonal *principal directions* of rotation, while the corresponding eigenvalues are the *moments of inertia*. Given the inertia tensor $T = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$, find the principal directions and moments of inertia.

Solution: Moments of inertia: $4, 2, 1$; principal directions: $(\,1, 2, 1\,)^T, (\,-1, 0, 1\,)^T, (\,1, -1, 1\,)^T$.

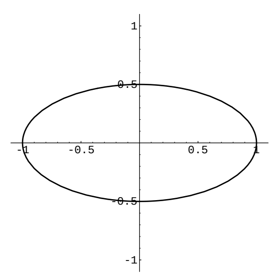
◇ 8.4.23. Let K be a positive definite 2×2 matrix. (*a*) Explain why the quadratic equation $\mathbf{x}^T K\,\mathbf{x} = 1$ defines an ellipse. Prove that its principal axes are the eigenvectors of K, and the semi-axes are the reciprocals of the square roots of the eigenvalues. (*b*) Graph and describe the following curves: (*i*) $x^2 + 4y^2 = 1$, (*ii*) $x^2 + xy + y^2 = 1$, (*iii*) $3x^2 + 2xy + y^2 = 1$. (*c*) What sort of curve(s) does $\mathbf{x}^T K\,\mathbf{x} = 1$ describe if K is not positive definite?

Solution:

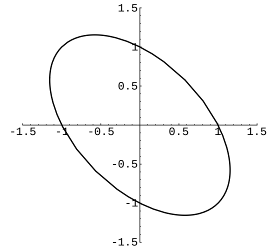
(a) Let $K = Q\Lambda Q^T$ be its spectral decomposition. Then $\mathbf{x}^T K\,\mathbf{x} = \mathbf{y}^T \Lambda\,\mathbf{y}$ where $\mathbf{x} = Q\,\mathbf{y}$. The ellipse $\mathbf{y}^T \Lambda\,\mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_1^2 = 1$ has principal axes aligned with the coordinate axes and semi-axes $1/\sqrt{\lambda_i}$, $i = 1, 2$. The map $\mathbf{x} = Q\,\mathbf{y}$ serves to rotate the coordinate axes to align with the columns of Q, i.e., the eigenvectors, while leaving the semi-axes unchanged.

(*b*)

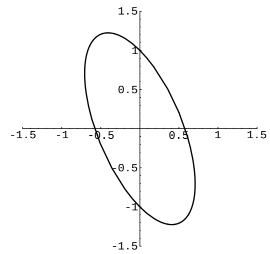
(*i*)



ellipse with semi-axes $1, \frac{1}{2}$ and principal axes $(\,1, 0\,)^T, (\,0, 1\,)^T$.

(ii) 

ellipse with semi-axes $\sqrt{2}, \sqrt{\frac{2}{3}}$,
and principal axes $(-1,1)^T, (1,1)^T$.

(iii) 

ellipse with semi-axes $\frac{1}{\sqrt{2+\sqrt{2}}}, \frac{1}{\sqrt{2-\sqrt{2}}}$,
and principal axes $\left(1+\sqrt{2}, 1\right)^T, \left(1-\sqrt{2}, 1\right)^T$.

(c) If K is symmetric and positive semi-definite it is a parabola; if K is indefinite, a hyperbola; if negative (semi-)definite, the empty set. If K is not symmetric, replace K by $\frac{1}{2}(K+K^T)$, and then apply the preceding classification.

♢ 8.4.24. Let K be a positive definite 3×3 matrix. (a) Prove that the quadratic equation $\mathbf{x}^T K \mathbf{x} = 1$ defines an ellipsoid in $\mathbb{R}^3$. What are its principal axes and semi-axes? (b) Describe the surface defined by the quadratic equation $11x^2 - 8xy + 20y^2 - 10xz + 8yz + 11z^2 = 1$.

Solution: (a) Same method as in Exercise 8.4.23. Its principal axes are the eigenvectors of K, and the semi-axes are the reciprocals of the square roots of the eigenvalues. (b) Ellipsoid with principal axes: $(1,0,1)^T, (-1,-1,1)^T, (-1,2,1)^T$ and semi-axes $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{12}}, \frac{1}{\sqrt{24}}$.

8.4.25. Prove that $A = A^T$ has a repeated eigenvalue if and only if it commutes, $AJ = JA$, with a nonzero skew-symmetric matrix: $J^T = -J \neq \mathrm{O}$. *Hint*: First prove this when A is a diagonal matrix.

Solution: If $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, then the (i,j) entry of ΛM is $d_i m_{ij}$, whereas the (i,j) entry of $M\Lambda$ is $d_k m_{ik}$. These are equal if and only if either $m_{ik} = 0$ or $d_i = d_k$. Thus, $\Lambda M = M\Lambda$ with M having one or more non-zero off-diagonal entries, which includes the case of non-zero skew-symmetric matrices, if and only if Λ has one or more repeated diagonal entries. Next, suppose $A = Q\Lambda Q^T$ is symmetric with diagonal form Λ. If $AJ = JA$, then $\Lambda M = M\Lambda$ where $M = Q^T J Q$ is also nonzero, skew-symmetric, and hence A has repeated eigenvalues. Conversely, if $\lambda_i = \lambda_j$, choose M such that $m_{ij} = 1 = -m_{ji}$, and then A commutes with $J = QMQ^T$. *Q.E.D.*

♢ 8.4.26. (a) Prove that every positive definite matrix K has a unique positive definite *square root*, i.e., a matrix $B > 0$ satisfying $B^2 = K$. (b) Find the positive definite square roots of the following matrices:

$$(i)\ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad (ii)\ \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}, \quad (iii)\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad (iv)\ \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}.$$

Solution:

(a) Set $B = Q\sqrt{\Lambda}\,Q^T$, where $\sqrt{\Lambda}$ is the diagonal matrix with the square roots of the eigenvalues of A along the diagonal. Uniqueness follows from the fact that the eigenvectors and eigenvalues are uniquely determined. (Permuting them does not change the final form of B.)

(b) (i) $\dfrac{1}{2}\begin{pmatrix} \sqrt{3}+1 & \sqrt{3}-1 \\ \sqrt{3}-1 & \sqrt{3}+1 \end{pmatrix}$; (ii) $\dfrac{1}{(2-\sqrt{2})\sqrt{2+\sqrt{2}}}\begin{pmatrix} 2\sqrt{2}-1 & 1-\sqrt{2} \\ 1-\sqrt{2} & 1 \end{pmatrix}$;

(iii) $\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 3 \end{pmatrix}$; (iv) $\begin{pmatrix} 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} & -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}} & -1+\frac{2}{\sqrt{3}} \\ -1+\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}} & 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}} & 1-\frac{2}{\sqrt{3}} \\ -1+\frac{2}{\sqrt{3}} & 1-\frac{2}{\sqrt{3}} & 1+\frac{4}{\sqrt{3}} \end{pmatrix}$.

8.4.27. Find all positive definite orthogonal matrices.

Solution: Only the identity matrix is both orthogonal and positive definite. Indeed, if $K = K^T > 0$ is orthogonal, then $K^2 = \mathrm{I}$, and so its eigenvalues are all ± 1. Positive definiteness implies that all the eigenvalues must be $+1$, and hence it diagonal form is $\Lambda = \mathrm{I}$. But then $K = Q\,\mathrm{I}\,Q^T = \mathrm{I}$ also.

♢ 8.4.28. *The Polar Decomposition*: Prove that every invertible matrix A has a *polar decomposition*, written $A = QB$, into the product of an orthogonal matrix Q and a positive definite matrix $B > 0$. Show that if $\det A > 0$, then Q is a proper orthogonal matrix. *Hint*: Look at the Gram matrix $K = A^T A$ and use Exercise 8.4.26. *Remark*: In mechanics, if A represents the deformation of a body, then Q represents a rotation, while B represents a stretching along the orthogonal eigendirections of K. Thus, any linear deformation of an elastic body can be decomposed into a pure stretching transformation followed by a rotation.

Solution: If $A = QB$, then $K = A^T A = B^T Q^T Q B = B^2$, and hence $B = \sqrt{K}$ is the positive definite square root of K. Moreover, $Q = AB^{-1}$ then satisfies $Q^T Q = B^{-T} A^T A B^{-1} = B^{-1} K B^{-1} = \mathrm{I}$ since $B^2 = K$. Finally, $\det A = \det Q \,\det B$ and $\det B > 0$ since $B > 0$. So if $\det A > 0$, then $\det Q = +1 > 0$. *Q.E.D.*

8.4.29. Find the polar decompositions $A = QB$, as defined in Exercise 8.4.28, of the following matrices:

(a) $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & -3 & 8 \\ 1 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix}$, (e) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$.

Solution: (a) $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}\begin{pmatrix} \sqrt{5} & 0 \\ 0 & 3\sqrt{5} \end{pmatrix}$,

(c) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$, (d) $\begin{pmatrix} 0 & -3 & 8 \\ 1 & 0 & 0 \\ 0 & 4 & 6 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{3}{5} & \frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{pmatrix}$,

(e) $\begin{pmatrix} 1 & 0 & 1 \\ 1 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} .3897 & .0323 & .9204 \\ .5127 & -.8378 & -.1877 \\ .7650 & .5450 & -.3430 \end{pmatrix}\begin{pmatrix} 1.6674 & -.2604 & .3897 \\ -.2604 & 2.2206 & .0323 \\ .3897 & .0323 & .9204 \end{pmatrix}$.

♡ 8.4.30. *The Spectral Decomposition*: (i) Let A be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let $P_k = \mathbf{u}_k \mathbf{u}_k^T$ be the orthogonal projection matrix onto the eigenline spanned by $\mathbf{u}_k$, as defined in Exercise 5.5.8. Prove that the spectral factorization (8.32) can be rewritten as

$$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_n P_n = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T, \tag{8.34}$$

expressing A as a linear combination of projection matrices. (ii) Write out the *spectral decomposition* (8.34) for the matrices in Exercise 8.4.13.

Solution:

(*i*) This follows immediately from the spectral factorization. The rows of ΛQ^T are $\lambda_1 \mathbf{u}_1^T, \ldots, \lambda_n \mathbf{u}_n^T$. The result then follows from the alternative version of matrix multiplication given in Exercise 1.2.32.

(*ii*) (a) $\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} = 5 \begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{pmatrix} - 5 \begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}.$

(*b*) $\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix} = (3+\sqrt{2}\,) \begin{pmatrix} \frac{3-2\sqrt{2}}{4-2\sqrt{2}} & \frac{1-\sqrt{2}}{4-2\sqrt{2}} \\ \frac{1-\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \end{pmatrix} + (3-\sqrt{2}\,) \begin{pmatrix} \frac{3+2\sqrt{2}}{4-2\sqrt{2}} & \frac{1+\sqrt{2}}{4-2\sqrt{2}} \\ \frac{1+\sqrt{2}}{4-2\sqrt{2}} & \frac{1}{4-2\sqrt{2}} \end{pmatrix}.$

(*c*) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3 \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$

(*d*) $\begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} = 4 \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$

♢ 8.4.31. *The Spectral Theorem for Hermitian Matrices*. Prove that any complex Hermitian matrix can be factored as $H = U \Lambda U^\dagger$ where U is a unitary matrix and Λ is a real diagonal matrix. *Hint*: See Exercises 5.3.25, 8.4.7.

Solution: ■

8.4.32. Find the spectral factorization, as in Exercise 8.4.31, of the following Hermitian matrices: (a) $\begin{pmatrix} 3 & 2\,\mathrm{i} \\ -2\,\mathrm{i} & 6 \end{pmatrix}$, (*b*) $\begin{pmatrix} 6 & 1-2\,\mathrm{i} \\ 1+2\,\mathrm{i} & 2 \end{pmatrix}$, (*c*) $\begin{pmatrix} -1 & 5\,\mathrm{i} & -4 \\ -5\,\mathrm{i} & -1 & 4\,\mathrm{i} \\ -4 & -4\,\mathrm{i} & 8 \end{pmatrix}$.

Solution:

(a) $\begin{pmatrix} 3 & 2\,\mathrm{i} \\ -2\,\mathrm{i} & 6 \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{i}}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2\,\mathrm{i}}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 7 \end{pmatrix} \begin{pmatrix} -\frac{\mathrm{i}}{\sqrt{5}} & \frac{2\,\mathrm{i}}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix},$

(*b*) $\begin{pmatrix} 6 & 1-2\,\mathrm{i} \\ 1+2\,\mathrm{i} & 2 \end{pmatrix} = \begin{pmatrix} \frac{1-2\,\mathrm{i}}{\sqrt{6}} & \frac{-1+2\,\mathrm{i}}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & \frac{\sqrt{5}}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 7 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+2\,\mathrm{i}}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ -\frac{1+2\,\mathrm{i}}{\sqrt{30}} & \frac{\sqrt{5}}{\sqrt{6}} \end{pmatrix},$

(*c*) $\begin{pmatrix} -1 & 5\,\mathrm{i} & -4 \\ -5\,\mathrm{i} & -1 & 4\,\mathrm{i} \\ -4 & -4\,\mathrm{i} & 8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{\mathrm{i}}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{\mathrm{i}}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 12 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{\mathrm{i}}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{\mathrm{i}}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{\mathrm{i}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$

8.4.33. Find the minimum and maximum values of the quadratic form $5x^2 + 4xy + 5y^2$ where x, y are subject to the constraint $x^2 + y^2 = 1$.

Solution: Maximum: 7; minimum: 3.

8.4.34. Find the minimum and maximum values of the quadratic form $2x^2 + xy + 2xz + 2y^2 + 2z^2$ where x, y, z are required to satisfy $x^2 + y^2 + z^2 = 1$.

Solution: Maximum: $\frac{4+\sqrt{5}}{2}$; minimum: $\frac{4-\sqrt{5}}{2}$.

8.4.35. Write down and solve a optimization principle characterizing the largest and smallest eigenvalue of the following positive definite matrices:

$$(a)\ \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \quad (b)\ \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}, \quad (c)\ \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}, \quad (d)\ \begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix}.$$

Solution:

(a) $\frac{5+\sqrt{5}}{2} = \max\{\, 2x^2 - 2xy + 3y^2 \mid x^2 + y^2 = 1 \,\}$,
$\frac{5-\sqrt{5}}{2} = \min\{\, 2x^2 - 2xy + 3y^2 \mid x^2 + y^2 = 1 \,\}$;

(b) $5 = \max\{\, 4x^2 + 2xy + 4y^2 \mid x^2 + y^2 = 1 \,\}$,
$3 = \min\{\, 4x^2 + 2xy + 4y^2 \mid x^2 + y^2 = 1 \,\}$;

(c) $12 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$,
$2 = \min\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$;

(d) $6 = \max\{\, 4x^2 - 2xy - 4xz + 4y^2 - 2yz + 4z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$,
$3 - \sqrt{3} = \min\{\, 4x^2 - 2xy - 4xz + 4y^2 - 2yz + 4z^2 \mid x^2 + y^2 + z^2 = 1 \,\}$.

8.4.36. Write down a maximization principle that characterizes the middle eigenvalue of the matrices in parts (*c*–*d*) of Exercise 8.4.35.

Solution: (*c*) $12 = \max\{\, 6x^2 - 8xy + 2xz + 6y^2 - 2yz + 11z^2 \mid x^2 + y^2 + z^2 = 1, x - y + 2z = 0 \,\}$;
(*d*) $3 + \sqrt{3} = \max\{\, 4x^2 - 2xy - 4xz + 4y^2 - 2yz + 4z^2 \mid x^2 + y^2 + z^2 = 1, x - z = 0 \,\}$,

8.4.37. What is the minimum and maximum values of the following rational functions:

$$(a)\ \frac{3x^2 - 2y^2}{x^2 + y^2}, \quad (b)\ \frac{x^2 - 3xy + y^2}{x^2 + y^2}, \quad (c)\ \frac{3x^2 + xy + 5y^2}{x^2 + y^2}, \quad (d)\ \frac{2x^2 + xy + 3xz + 2y^2 + 2z^2}{x^2 + y^2 + z^2}.$$

Solution:

(a) Maximum: 3; minimum: -2.

(b) Maximum: $\frac{5}{2}$; minimum: $-\frac{1}{2}$.

(c) Maximum: $\frac{8+\sqrt{5}}{2} = 5.11803$; minimum: $\frac{8-\sqrt{5}}{2} = 2.88197$.

(d) Maximum: $\frac{4+\sqrt{10}}{2} = 3.58114$; minimum: $\frac{4-\sqrt{10}}{2} = .41886$.

8.4.38. Find the minimum and maximum values of $q(\mathbf{x}) = 2 \sum_{i=1}^{n-1} x_i\, x_{i+1}$ for $\|\mathbf{x}\|^2 = 1$. *Hint*: See Exercise 8.2.46.

Solution: Maximum: $2\cos\dfrac{\pi}{n+1}$; minimum $-2\cos\dfrac{\pi}{n+1}$.

8.4.39. Suppose $K > 0$. What is the maximum value of $q(\mathbf{x}) = \mathbf{x}^T K\, \mathbf{x}$ when $\mathbf{x}$ is constrained to a sphere of radius $\|\mathbf{x}\| = r$?

Solution: Maximum: $r^2 \lambda_1$; minimum $r^2 \lambda_n$, where λ_1, λ_n are, respectively, the maximum and minimum eigenvalues of K.

8.4.40. Let $K > 0$. Prove the product formula

$$\max\left\{\, \mathbf{x}^T K\, \mathbf{x} \;\middle|\; \|\mathbf{x}\| = 1 \,\right\}\ \min\left\{\, \mathbf{x}^T K^{-1} \mathbf{x} \;\middle|\; \|\mathbf{x}\| = 1 \,\right\} = 1.$$

Solution: $\max\{\, \mathbf{x}^T K\, \mathbf{x} \mid \|\mathbf{x}\| = 1 \,\} = \lambda_1$ is the largest eigenvalue of K. On the other hand, K^{-1} is positive definite, cf. Exercise 3.4.9, and hence $\min\{\, \mathbf{x}^T K^{-1} \mathbf{x} \mid \|\mathbf{x}\| = 1 \,\} = \mu_n$ is its smallest eigenvalue. But the eigenvalues of K^{-1} are the reciprocals of the eigenvalues of K, and hence its smallest eigenvalue is $\mu_n = 1/\lambda_1$, and so the product is $\lambda_1 \mu_n = 1$. *Q.E.D.*

◇ 8.4.41. Write out the details in the proof of Theorem 8.30.

Solution: According to the discussion preceding the statement of the Theorem 8.30,
$$\lambda_j = \max\left\{\, \mathbf{y}^T \Lambda\, \mathbf{y} \;\middle|\; \|\,\mathbf{y}\,\| = 1,\ \mathbf{y}\cdot\mathbf{e}_1 = \cdots = \mathbf{y}\cdot\mathbf{e}_{j-1} = 0 \,\right\}.$$
Moreover, using (8.33), setting $\mathbf{x} = Q\mathbf{y}$ and using the fact that Q is an orthogonal matrix and so $(Q\mathbf{v})\cdot(Q\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$ for any $\mathbf{v},\mathbf{w} \in \mathbb{R}^n$, we have
$$\mathbf{x}^T A\,\mathbf{x} = \mathbf{y}^T \Lambda\,\mathbf{y}, \qquad \|\,\mathbf{x}\,\| = \|\,\mathbf{y}\,\|, \qquad \mathbf{y}\cdot\mathbf{e}_i = \mathbf{x}\cdot\mathbf{v}_i,$$
where $\mathbf{v}_i = Q\,\mathbf{e}_i$ is the i^{th} eigenvector of A. Therefore, by the preceding formula,
$$\lambda_j = \max\left\{\, \mathbf{x}^T A\,\mathbf{x} \;\middle|\; \|\,\mathbf{x}\,\| = 1,\ \mathbf{x}\cdot\mathbf{v}_1 = \cdots = \mathbf{x}\cdot\mathbf{v}_{j-1} = 0 \,\right\}. \qquad \textit{Q.E.D.}$$

◇ 8.4.42. Reformulate Theorem 8.30 as a minimum principle for intermediate eigenvalues.

Solution: Let A be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and corresponding orthogonal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. Then the minimal value of the quadratic form $\mathbf{x}^T A\,\mathbf{x}$ over all unit vectors which are orthogonal to the last $n-j$ eigenvectors is the j^{th} eigenvalue:
$$\lambda_j = \min\left\{\, \mathbf{x}^T A\,\mathbf{x} \;\middle|\; \|\,\mathbf{x}\,\| = 1,\quad \mathbf{x}\cdot\mathbf{v}_{j+1} = \cdots = \mathbf{x}\cdot\mathbf{v}_n = 0 \,\right\}.$$

8.4.43. Under the set-up of Theorem 8.30, explain why
$$\lambda_j = \max\left\{\, \frac{\mathbf{v}^T K \mathbf{v}}{\|\,\mathbf{v}\,\|^2} \;\middle|\; \mathbf{v} \neq \mathbf{0},\quad \mathbf{v}\cdot\mathbf{v}_1 = \cdots = \mathbf{v}\cdot\mathbf{v}_{j-1} = 0 \,\right\}.$$

Solution: Note that $\dfrac{\mathbf{v}^T K \mathbf{v}}{\|\,\mathbf{v}\,\|^2} = \mathbf{u}^T K\,\mathbf{u}$, where $\mathbf{u} = \dfrac{\mathbf{v}}{\|\,\mathbf{v}\,\|}$ is a unit vector. Moreover, if $\mathbf{v}$ is orthogonal to an eigenvector $\mathbf{v}_i$, so is $\mathbf{u}$. Therefore,
$$\max\left\{\, \frac{\mathbf{v}^T K \mathbf{v}}{\|\,\mathbf{v}\,\|^2} \;\middle|\; \mathbf{v} \neq \mathbf{0},\quad \mathbf{v}\cdot\mathbf{v}_1 = \cdots = \mathbf{v}\cdot\mathbf{v}_{j-1} = 0 \,\right\}$$
$$= \max\left\{\, \mathbf{u}^T K\,\mathbf{u} \;\middle|\; \|\,\mathbf{u}\,\| = 1,\quad \mathbf{u}\cdot\mathbf{v}_1 = \cdots = \mathbf{u}\cdot\mathbf{v}_{j-1} = 0 \,\right\} = \lambda_j$$
by Theorem 8.30.

♡ 8.4.44. (*a*) Let K, M be positive definite $n \times n$ matrices and $\lambda_1 \geq \cdots \geq \lambda_n$ their generalized eigenvalues, as in Exercise 8.4.9. Prove that that the largest generalized eigenvalue can be characterized by the maximum principle $\lambda_1 = \max\{\, \mathbf{x}^T K\,\mathbf{x} \mid \mathbf{x}^T M\,\mathbf{x} = 1 \,\}$. *Hint*: Use Exercise 8.4.26. (*b*) Prove the alternative maximum principle $\lambda_1 = \max\left\{\, \dfrac{\mathbf{x}^T K\,\mathbf{x}}{\mathbf{x}^T M\,\mathbf{x}} \;\middle|\; \mathbf{x} \neq \mathbf{0} \,\right\}$. (*c*) How would you characterize the smallest generalized eigenvalue? (*d*) An intermediate generalized eigenvalue?

Solution:

(*a*) Let $R = \sqrt{M}$ be the positive definite square root of M, and set $\widehat{K} = R^{-1} K\, R^{-1}$. Then $\mathbf{x}^T K\,\mathbf{x} = \mathbf{y}^T \widetilde{K}\,\mathbf{y}$, $\mathbf{x}^T M\,\mathbf{x} = \mathbf{y}^T\mathbf{y} = \|\,\mathbf{y}\,\|^2$, where $\mathbf{y} = R\mathbf{x}$. Thus,
$$\max\left\{\, \mathbf{x}^T K\,\mathbf{x} \;\middle|\; \mathbf{x}^T M\,\mathbf{x} = 1 \,\right\} = \max\left\{\, \mathbf{y}^T \widetilde{K}\,\mathbf{y} \;\middle|\; \|\,\mathbf{y}\,\|^2 = 1 \,\right\} = \widetilde{\lambda}_1,$$
the largest eigenvalue of $\widetilde{K}$. But $\widetilde{K}\,\mathbf{y} = \lambda\mathbf{y}$ implies $K\,\mathbf{x} = \lambda\,\mathbf{x}$, and so the eigenvalues of K and $\widetilde{K}$ coincide. *Q.E.D.*

(*b*) Write $\mathbf{y} = \dfrac{\mathbf{x}}{\sqrt{\mathbf{x}^T M\,\mathbf{x}}}$ so that $\mathbf{y}^T M\,\mathbf{y} = 1$. Then, by part (*a*),
$$\max\left\{\, \frac{\mathbf{x}^T K\,\mathbf{x}}{\mathbf{x}^T M\,\mathbf{x}} \;\middle|\; \mathbf{x} \neq \mathbf{0} \,\right\} = \max\left\{\, \mathbf{y}^T K\,\mathbf{y} \;\middle|\; \mathbf{y}^T M\,\mathbf{y} = 1 \,\right\} = \lambda_1.$$

(c) $\lambda_n = \min\{\,\mathbf{x}^T K\,\mathbf{x} \;\mid\; \mathbf{x}^T M\,\mathbf{x} = 1\,\}$.

(d) $\lambda_j = \max\{\,\mathbf{x}^T K\,\mathbf{x} \;\mid\; \mathbf{x}^T M\,\mathbf{x} = 1,\ \mathbf{x}^T M\,\mathbf{v}_1 = \cdots = \mathbf{x}^T M\,\mathbf{v}_{j-1} = 0\,\}$ where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are the generalized eigenvectors.

8.4.45. Use Exercise 8.4.44 to find the minimum and maximum of the rational functions (a) $\dfrac{3x^2+2y^2}{4x^2+5y^2}$, ($b$) $\dfrac{x^2-xy+2y^2}{2x^2-xy+y^2}$, ($c$) $\dfrac{2x^2+3y^2+z^2}{x^2+3y^2+2z^2}$, ($d$) $\dfrac{2x^2+6xy+11y^2+6yz+2z^2}{x^2+2xy+3y^2+2yz+z^2}$.

Solution: (a) $\max = \frac{3}{4}$, $\min = \frac{2}{5}$; (b) $\max = \frac{9+4\sqrt{2}}{7}$, $\min = \frac{9-4\sqrt{2}}{7}$;
(c) $\max = 2$, $\min = \frac{1}{2}$; (d) $\max = 4$, $\min = 1$.

8.4.46. Let A be a complete square matrix, not necessarily symmetric, with all positive eigenvalues. Is the quadratic form $q(\mathbf{x}) = \mathbf{x}^T A\,\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$?

Solution: No. For example, if $A = \begin{pmatrix} 1 & b \\ 0 & 4 \end{pmatrix}$, then $q(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ if and only if $|\,b\,| < 4$.

SVD
8.5. Singular Values.

8.5.1. Find the singular values of the following matrices: (a) $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, ($b$) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, ($c$) $\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix}$, ($d$) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$, ($e$) $\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$, ($f$) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$.

Solution: (a) $3 \pm \sqrt{5}$, (b) $1, 1$; (c) $5\sqrt{2}$; (d) $3, 2$; (e) $\sqrt{7}, \sqrt{2}$; (f) $3, 1$.

8.5.2. Write out the singular value decomposition (8.40) of the matrices in Exercise 8.5.1.

Solution:

(a) $$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}} & \frac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \frac{2}{\sqrt{10-2\sqrt{5}}} & \frac{2}{\sqrt{10+2\sqrt{5}}} \end{pmatrix} \begin{pmatrix} \sqrt{3+\sqrt{5}} & 0 \\ 0 & \sqrt{3-\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{-2+\sqrt{5}}{\sqrt{10-4\sqrt{5}}} & \frac{1}{\sqrt{10-4\sqrt{5}}} \\ \frac{-2-\sqrt{5}}{\sqrt{10+4\sqrt{5}}} & \frac{1}{\sqrt{10+4\sqrt{5}}} \end{pmatrix},$$

(b) $$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(c) $$\begin{pmatrix} 1 & -2 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \begin{pmatrix} 5\sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix},$$

(d) $$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

(e) $$\begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{4}{\sqrt{35}} & -\frac{3}{\sqrt{35}} & \frac{1}{\sqrt{35}} & \frac{3}{\sqrt{35}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix},$$

(f) $$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0\frac{1}{\sqrt{2}} & \end{pmatrix}.$$

8.5.3. (*a*) Construct the singular value decomposition of the shear matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(*b*) Explain how a shear can be realized as a combination of a rotation, a stretch, followed by a second rotation.

Solution:

(a) $$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1+\sqrt5}{\sqrt{10+2\sqrt5}} & \frac{1-\sqrt5}{\sqrt{10-2\sqrt5}} \\ \frac{2}{\sqrt{10+2\sqrt5}} & \frac{2}{\sqrt{10-2\sqrt5}} \end{pmatrix} \begin{pmatrix} \sqrt{\frac32 + \frac12\sqrt5} & 0 \\ 0 & \sqrt{\frac32 - \frac12\sqrt5} \end{pmatrix} \begin{pmatrix} \frac{-1+\sqrt5}{\sqrt{10-2\sqrt5}} & \frac{2}{\sqrt{10-2\sqrt5}} \\ \frac{-1-\sqrt5}{\sqrt{10+2\sqrt5}} & \frac{2}{\sqrt{10+2\sqrt5}} \end{pmatrix};$$

(*b*) The first and last matrices are proper orthogonal, and so represent rotations, while the middle matrix is a stretch along the coordinate directions, in proportion to the singular values. Matrix multiplication corresponds to composition of the corresponding linear transformations.

8.5.4. Find the condition number of the following matrices. Which would you characterize as ill-conditioned? (*a*) $\begin{pmatrix} 2 & -1 \\ -3 & 1 \end{pmatrix}$, (*b*) $\begin{pmatrix} -.999 & .341 \\ -1.001 & .388 \end{pmatrix}$, (*c*) $\begin{pmatrix} 1 & 2 \\ 1.001 & 1.9997 \end{pmatrix}$,

(*d*) $\begin{pmatrix} -1 & 3 & 4 \\ 2 & 10 & 6 \\ 1 & 2 & -3 \end{pmatrix}$, (*e*) $\begin{pmatrix} 72 & 96 & 103 \\ 42 & 55 & 59 \\ 67 & 95 & 102 \end{pmatrix}$, (*f*) $\begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix}$.

Solution:

(*a*) The eigenvalues of $K = A^T A$ are $\frac{15}{2} \pm \frac{\sqrt{221}}{2} = 14.933, .0669656$. The square roots of these eigenvalues give us the singular values of A. i.e., $3.8643, .2588$. The condition number is $3.86433 \,/\, .25878 = 14.9330$.

(*b*) The singular values are $1.50528, .030739$, and so the condition number is $1.50528 \,/\, .030739 = 48.9697$.

(*c*) The singular values are $3.1624, .0007273$, and so the condition number is $3.1624 \,/\, .0007273 = 4348.17$; slightly ill-conditioned.

(*d*) The singular values are $12.6557, 4.34391, .98226$, so he condition number is $12.6557 \,/\, .98226 = 12.88418$.

(*e*) The singular values are $239.138, 3.17545, .00131688$, so the condition number is $239.138 \,/\, .00131688 = 181594$; ill-conditioned.

(*f*) The singular values are $30.2887, 3.85806, .843107, .01015$, so the condition number is $30.2887 \,/\, .01015 = 2984.09$; slightly ill-conditioned.

♠ 8.5.5. Solve the following systems of equations using Gaussian Elimination with three-digit rounding arithmetic. Is your answer a reasonable approximation to the exact solution? Compare the accuracy of your answers with the condition number of the coefficient matrix, and discuss the implications of ill-conditioning.

(a) $\begin{aligned} 1000\,x + 999\,y &= 1, \\ 1001\,x + 1002\,y &= -1, \end{aligned}$ (*b*) $\begin{aligned} 97\,x + 175\,y + 83\,z &= 1, \\ 44\,x + 78\,y + 37\,z &= 1, \\ 52\,x + 97\,y + 46\,z &= 1. \end{aligned}$ (*c*) $\begin{aligned} 3.001\,x + 2.999\,y + 5\,z &= 1, \\ -x + 1.002\,y - 2.999\,z &= 2 \\ 2.002\,x + 4\,y + 2\,z &= 1.002. \end{aligned}$

Solution:

(*a*) $x = 1, y = -1$.

(*b*) $x = -1, y = -109, z = 231$, singular values $26.6, 1.66, .0023$, condition number: 1.17×10^5.

(*c*) ∎

♠ 8.5.6. (*a*) Compute the singular values and condition numbers of the 2×2, 3×3, and 4×4 Hilbert matrices. (*b*) What is the smallest Hilbert matrix with condition number larger than 10^6?

Solution:

(a) The 2×2 Hilbert matrix has singular values $1.2676, .0657$ and condition number 19.2815. The 3×3 Hilbert matrix has singular values $1.4083, .1223, .0027$ and condition number 524.057. The 4×4 Hilbert matrix has singular values $1.5002, .1691, .006738, .0000967$ and condition number 15513.7.

(b) An 5×5 Hilbert matrix has condition number 4.7661×10^5. An 6×6 Hilbert matrix has condition number 1.48294×10^7.

8.5.7. (a) What are the singular values of a $1 \times n$ matrix? (b) Write down its singular value decomposition. (c) Write down its pseudoinverse.

Solution: Let $A = \mathbf{v} \in \mathbb{R}^n$ be the matrix (column vector) in question. (a) It has one singular value: $\| \mathbf{v} \|$; (b) $P = \dfrac{\mathbf{v}}{\| \mathbf{v} \|}$, $\Sigma = \big(\| \mathbf{v} \| \big)$, $Q = (1)$; (c) $\mathbf{v}^+ = \dfrac{\mathbf{v}^T}{\| \mathbf{v} \|^2}$.

8.5.8. Answer Exercise 8.5.7 for an $m \times 1$ matrix.

Solution: Let $A = \mathbf{v}^T$, where $\mathbf{v} \in \mathbb{R}^n$, be the matrix (row vector) in question. (a) It has one singular value: $\| \mathbf{v} \|$; (b) $P = (1)$, $\Sigma = \big(\| \mathbf{v} \| \big)$, $Q = \dfrac{\mathbf{v}}{\| \mathbf{v} \|}$; (c) $\mathbf{v}^+ = \dfrac{\mathbf{v}}{\| \mathbf{v} \|^2}$.

8.5.9. *True or false*: Every matrix has at least one singular value.

Solution: True, with one exception — the zero matrix.

8.5.10. Explain why the singular values of A are the same as the nonzero eigenvalues of the positive definite square root matrix $S = \sqrt{A^T A}$, defined in Exercise 8.4.26.

Solution: Since $S^2 = K = A^T A$, the eigenvalues λ of K are the squares, $\lambda = \sigma^2$ of the eigenvalues σ of S. Moreover, since $S > 0$, its eigenvalues are all non-negative, so $\sigma = + \sqrt{\lambda}$, and, by definition, the nonzero $\sigma > 0$ are the singular values of A.

8.5.11. *True or false*: The singular values of A^T are the same as the singular values of A.

Solution: True. If $A = P \Sigma Q^T$ is the singular value decomposition of A, then the transposed equation $A^T = Q \Sigma P^T$ gives the singular value decomposition of A^T, and so the diagonal entries of Σ are also the singular values of A^T.

♢ 8.5.12. Prove that if A is square, nonsingular, then the singular values of A^{-1} are the reciprocals of the singular values of A. How are their condition numbers related?

Solution: Since A is nonsingular, so is $K = A^T A$, and hence all its eigenvalues are nonzero. Thus, Q, whose columns are the orthonormal eigenvector basis of K, is a square orthogonal matrix, as is P. Therefore, the singular value decomposition of the inverse matrix is $A^{-1} = Q^{-T} \Sigma^{-1} P^{-1} = Q \Sigma^{-1} P^T$. The diagonal entries of Σ^{-1}, which are the singular values of A^{-1}, are the reciprocals of the diagonal entries of Σ. Finally, $\kappa(A^{-1}) = \sigma_n / \sigma_1 = 1/\kappa(A)$. *Q.E.D.*

♢ 8.5.13. Let A be any matrix. Prove that any vector $\mathbf{x}$ that minimizes the Euclidean least squares error $\| A\mathbf{x} - \mathbf{b} \|$ must satisfy the normal equations $A^T A \mathbf{x} = A^T \mathbf{b}$, even when $\ker A \neq \{\mathbf{0}\}$.

Solution: ■

♢ 8.5.14. (a) Let A be a nonsingular square matrix. Prove that the product of the singular values of A equals the absolute value of its determinant: $\sigma_1 \sigma_2 \cdots \sigma_n = |\det A|$. (b) Does their sum equal the absolute value of the trace: $\sigma_1 + \cdots + \sigma_n = |\operatorname{tr} A|$? (c) Show that if $|\det A| < 10^{-k}$, then its minimal singular value satisfies $\sigma_n < 10^{-k/n}$. (d) *True or false*: A matrix whose determinant is very small is ill-conditioned. (e) Construct an ill-conditioned matrix with $\det A = 1$.

Solution:

(a) Since all matrices in its singular value decomposition (8.40) are square, we can take the determinant as follows:

$$\det A = \det P \ \det \Sigma \ \det Q^T = \pm 1 \ \det \Sigma = \pm \sigma_1 \sigma_2 \ \cdots \ \sigma_n,$$

since the determinant of an orthogonal matrix is ± 1. The result follows upon taking absolute values of this equation and using the fact that the product of the singular values is non-negative.

(b) No — even simple nondiagonal examples show this is false.

(c) Numbering the singular values in decreasing order, so $\sigma_k \geq \sigma_n$ for all k, we conclude $10^{-k} > |\det A| = \sigma_1 \sigma_2 \ \cdots \ \sigma_n \geq \sigma_n^n$, and the result follows by taking the n^{th} root.

(d) Not necessarily, since all the singular values could be very small but equal, and so the condition number would be 1.

(e) The diagonal matrix with entries 10^k and 10^{-k} for $k \gg 0$, or more generally, any 2×2 matrix with singular values 10^k and 10^{-k}, has condition number 10^{2k}.

8.5.15. *True or false*: If $\det A > 1$, then A is not ill-conditioned.

Solution: False. For example, the diagonal matrix with entries $2 \cdot 10^k$ and 10^{-k} for $k \gg 0$ has determinant 2 but condition number $2 \cdot 10^{2k}$.

8.5.16. *True or false*: If A is a symmetric matrix, then its singular values are the same as its eigenvalues.

Solution: False — the singular values are the absolute values of the nonzero eigenvalues.

8.5.17. *True or false*: If U is an upper triangular matrix whose diagonal entries are all positive, then its singular values are the same as its diagonal entries.

Solution: False. For example, $U = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has singular values $3 \pm \sqrt{5}$.

8.5.18. *True or false*: The singular values of A^2 are the squares σ_i^2 of the singular values of A.

Solution: False, unless A is symmetric or, more generally, normal, meaning that $A^T A = A A^T$. For example, the singular values of $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are $\sqrt{\frac{3}{2} \pm \frac{\sqrt{5}}{2}}$, while the singular values of $A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ are $\sqrt{3 \pm 2\sqrt{2}}$.

8.5.19. *True or false*: If $B = S^{-1} A S$ are similar matrices, then A amd B have the same singular values.

Solution: False. This is only true if S is an orthogonal matrix.

◇ 8.5.20. A complex matrix A is called *normal* if it commutes with its Hermitian transpose $A^\dagger = \overline{A^T}$, as defined in so $A^\dagger A = A A^\dagger$.(a) Show that a real matrix is normal if it commutes with its transpose: $A^T A = A A^T$. (b) Show that every real symmetric matrix is normal. (c) Find a normal matrix which is real but not symmetric. (d) Show that the eigenvectors of a normal matrix form an orthogonal basis of $\mathbb{C}^n$ under the Hermitian dot product. (e) Show that the converse is true: a matrix has an orthogonal eigenvector basis of $\mathbb{C}^n$ if and only if it is normal. (f) Prove that if A is normal, the singular values of A^n are σ_i^n where σ_i are the singular values of A. Show that this result is not true if A is not normal.

Solution:

(a) If A is real, $A^\dagger = A^T$, and so the result follows immediately.

(b) If $A = A^T$ then $A^T A = A^2 = A A^T$.

(c) $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, or more generally, $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ where $b \neq 0$, cf. Exercise 1.6.7.

(d) ■

(e) Let $U = (\, \mathbf{u}_1 \; \mathbf{u}_2 \; \ldots \; \mathbf{u}_n \,)$ be the correspponding unitary matrix, with $U^{-1} = U^\dagger$. Then $A U = U \Lambda$, where Λ is the diagonal eigenvalue matrix, and so $A = U \Lambda U^\dagger = U \Lambda \overline{U}^T$. Then $A A^T = U \Lambda \overline{U}^T \overline{U} \Lambda U^T$ ■

♡ 8.5.21. Let A be a nonsingular 2×2 matrix with singular value decomposition $A = P \Sigma Q^T$ and singular values $\sigma_1 \geq \sigma_2 > 0$. (*a*) Prove that the image of the unit (Euclidean) circle under the linear transformation defined by A is an ellipse, $E = \{\, A\mathbf{x} \mid \| \mathbf{x} \| = 1 \,\}$, whose principal axes are the columns $\mathbf{p}_1, \mathbf{p}_2$ of P, and whose corresponding semi-axes are the singular values σ_1, σ_2. (*b*) Show that if A is symmetric, then the ellipse's principal axes are the eigenvectors of A and the semi-axes are the absolute values of its eigenvalues. (*c*) Prove that the area of E equals $\pi \,|\det A\,|$. (*d*) Find the principal axes, semi-axes, and area of the ellipses defined by: (*i*) $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, (*ii*) $\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, (*iii*) $\begin{pmatrix} 5 & -4 \\ 0 & -3 \end{pmatrix}$. (*e*) What happens if A is singular?

Solution:

(a) If $\| \mathbf{x} \| = 1$, then $\mathbf{y} = A\mathbf{x}$ satisfies the equation $\mathbf{y}^T B \mathbf{y} = 1$, where $B = A^{-T} A^{-1} = P^T \Sigma^{-2} P$. Thus, by Exercise 8.4.23, the principal axes of the ellipse are the columns of P, and the semi-axes are the reciprocals of the square roots of the diagonal entries of Σ^{-2}, which are precisely the singular values σ_i.

(b) If A is symmetric, $P = Q$ is the orthogonal eigenvector matrix, and so the columns of P coincide with the eigenvectors of A. Moreover, the singular values $\sigma_i = |\,\lambda_i\,|$ are the absolute values of its eigenvalues.

(c) From elementary geometry, the area of an ellipse equals π times the product of its semi-axes. Thus, area$E = \pi\, \sigma_1 \sigma_2 = \pi \,|\det A\,|$ using Exercise 8.5.14.

(d) (*i*) principal axes: $\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix}$; semi-axes: $\sqrt{\frac{3}{2} + \frac{\sqrt{5}}{2}}, \sqrt{\frac{3}{2} - \frac{\sqrt{5}}{2}}$; area: π.

(*ii*) principal axes: $\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (but any orthogonal basis of $\mathbb{R}^2$ will also do); semi-axes: $\sqrt{5}, \sqrt{5}$ (circle); area: 5π.

(*iii*) principal axes: $\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}$; semi-axes: $3\sqrt{5}\,, \sqrt{5}\,$; area: 15π.

(*iv*) If $A = \mathrm{O}$, then $E = \{\mathbf{0}\}$ is a point. Otherwise, rank $A = 1$ and its singular value decomposition is $A = \sigma_1 \mathbf{p}_1 \mathbf{q}_1^T$ where $A\mathbf{q}_1 = \sigma_1 \mathbf{p}_1$. Then E is a line segment in the direction of $\mathbf{p}_1$ whose length is $2\sigma_1$.

8.5.22. Let $A = \begin{pmatrix} 2 & -3 \\ -3 & 10 \end{pmatrix}$. Write down the equation for the ellipse $E = \{\, A\mathbf{x} \mid \| \mathbf{x} \| = 1 \,\}$ and draw a picture. What are its principal axes? Its semi-axes? Its area?

Solution: $\left(\frac{10}{11} u + \frac{3}{11} v \right)^2 + \left(\frac{3}{11} u + \frac{2}{11} v \right)^2 = 1$ or $109 u^2 + 72 u v + 13 v^2 = 121$. Since A is symmetric, the semi-axes are the eigenvalues, which are the same as the singular values, namely $11, 1$; the principal axes are the eigenvectors, $\begin{pmatrix} -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; the area is 11π. ■ell1■

8.5.23. Let $A = \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}$, and let $E = \{ \mathbf{y} = A\mathbf{x} \mid \| \mathbf{x} \| = 1 \}$ be the image of the unit Euclidean sphere under the linear map induced by A. (*a*) Explain why E is an ellipsoid and write down its equation. (*b*) What are its principal axes and their lengths — the semi-axes of the ellipsoid? (*c*) What is the volume of the solid ellipsoidal domain enclosed by E?

Solution:

(*a*) In view of the singular value decomposition of $A = P \Sigma Q^T$, the set E is obtained by first rotating the unit sphere according to Q^T, which doesn't change it, then stretching it along the coordinate axes into an ellipsoid according to Σ, and then rotating it according to P, which aligns its principal axes with the columns of P. The equation is

$$(65u + 43v - 2w)^2 + (43u + 65v + 2w)^2 + (-2u + 2v + 20w)^2 = 216^2, \qquad \text{or}$$
$$1013u^2 + 1862uv + 1013v^2 - 28uw + 28vw + 68w^2 = 7776.$$

(*b*) The semi-axes are the eigenvalues: $12, 9, 2$; the principal axes are the eigenvectors: $(1, -1, 2)^T, \ (-1, 1, 1)^T, \ (1, 1, 0)^T$.

(*c*) Since the unit sphere has volume $\frac{4}{3}\pi$, the volume of E is $\frac{4}{3}\pi \det A = 288\pi$.

♢ 8.5.24. *Optimization Principles for Singular Values*: Let A be any nonzero $m \times n$ matrix. Prove that (*a*) $\sigma_1 = \max\{ \| A\mathbf{u} \| \mid \| \mathbf{u} \| = 1 \}$. (*b*) Is the minimum the smallest singular value? (*c*) Can you design an optimization principle for the intermediate singular values?

Solution:

(*a*) $\| A\mathbf{u} \|^2 = (A\mathbf{u})^T A\mathbf{u} = \mathbf{u}^T K \mathbf{u}$, where $K = A^T A$. According to Theorem 8.28, $\max\{ \| A\mathbf{u} \|^2 = \mathbf{u}^T K \mathbf{u} \mid \| \mathbf{u} \| = 1 \}$ is the largest eigenvalue λ_1 of $K = A^T A$, hence the maximum value of $\| A\mathbf{u} \|$ is $\sqrt{\lambda_1} = \sigma_1$. *Q.E.D.*

(*b*) This is true if $\operatorname{rank} A = n$ by the same reasoning, but false if $\ker A \neq \{\mathbf{0}\}$, since then the minimum is 0, but, according to our definition, singular values are always nonzero.

(*c*) The k^{th} singular value σ_k is obtained by maximizing $\| A\mathbf{u} \|$ over all unit vectors which are orthogonal to the first $k-1$ singular vectors.

♢ 8.5.25. Let A be a square matrix. Prove that its maximal eigenvalue is smaller than its maximal singular value: $\max |\lambda_i| \leq \max \sigma_i$. *Hint*: Use Exercise 8.5.24.

Solution: Let λ_1 be the maximal eigenvalue, and let $\mathbf{u}_1$ be a corresponding unit eigenvector. By Exercise 8.5.24, $\sigma_1 \geq \| A\mathbf{u}_1 \| = |\lambda_1|$.

8.5.26. Let A be a nonsingular square matrix. Prove that $\kappa(A) = \dfrac{\max\{ \| A\mathbf{u} \| \mid \| \mathbf{u} \| = 1 \}}{\min\{ \| A\mathbf{u} \| \mid \| \mathbf{u} \| = 1 \}}$.

Solution: By Exercise 8.5.24, the numerator is the largest singular value, while the denominator is the smallest, and so the ratio is the condition number.

8.5.27. Find the pseudoinverse of the following matrices: (*a*) $\begin{pmatrix} 1 & -1 \\ -3 & 3 \end{pmatrix}$, (*b*) $\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$, (*c*) $\begin{pmatrix} 2 & 0 \\ 0 & -1 \\ 0 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (*e*) $\begin{pmatrix} 1 & -1 & 1 \\ -2 & 2 & -2 \end{pmatrix}$, (*f*) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \\ 3 & 9 \end{pmatrix}$, (*g*) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.

Solution: (a) $\begin{pmatrix} \frac{1}{20} & -\frac{3}{20} \\ -\frac{1}{20} & \frac{3}{20} \end{pmatrix}$, (*b*) $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{pmatrix}$, (*c*) $\begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$,

$$(e)\ \begin{pmatrix} \frac{1}{15} & -\frac{2}{15} \\ -\frac{1}{15} & \frac{2}{15} \\ \frac{1}{15} & -\frac{2}{15} \end{pmatrix}, \qquad (f)\ \begin{pmatrix} \frac{1}{140} & \frac{1}{70} & \frac{3}{140} \\ \frac{3}{140} & \frac{3}{70} & \frac{9}{140} \end{pmatrix}, \qquad (g)\ \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} & \frac{2}{9} \\ \frac{5}{18} & \frac{2}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{4}{9} & -\frac{7}{18} \end{pmatrix}.$$

8.5.28. Use the pseudoinverse to find the least squares solution of minimal norm to the following linear systems: (a) $\begin{aligned} x+y&=1, \\ 3x+3y&=-2. \end{aligned}$ (b) $\begin{aligned} x-3y&=2, \\ 2x+y&=-1, \\ x+y&=0. \end{aligned}$ (c) $\begin{aligned} x+y+z&=5, \\ 2x-y+z&=2. \end{aligned}$

Solution:

(a) $A = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}, \quad A^+ = \begin{pmatrix} \frac{1}{20} & \frac{3}{20} \\ \frac{1}{20} & \frac{3}{20} \end{pmatrix}, \quad \mathbf{x}^\star = A^+ \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix};$

(b) $A = \begin{pmatrix} 1 & -3 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A^+ = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{3}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}, \quad \mathbf{x}^\star = A^+ \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{7}{11} \end{pmatrix};$

(c) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \end{pmatrix}, \quad A^+ = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{5}{14} \\ \frac{2}{7} & \frac{1}{14} \end{pmatrix}, \quad \mathbf{x}^\star = A^+ \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{9}{7} \\ \frac{15}{7} \\ \frac{11}{7} \end{pmatrix}.$

♡ 8.5.29. Prove that the pseudoinverse satisfies the following identities: (a) $(A^+)^+ = A$, (b) $A A^+ A = A$, (c) $A^+ A A^+ = A^+$, (d) $(A A^+)^T = A A^+$, (e) $(A^+ A)^T = A^+ A$.

Solution: We repeatedly use the fact that the columns of P, Q are orthonormal, and so $P^T P = \mathrm{I}, Q^T Q = \mathrm{I}$.

(a) Since $A^+ = Q\,\Sigma^{-1} P^T$ is the singular value decomposition of A^+, we have $(A^+)^+ = P\,(\Sigma^{-1})^{-1} Q^T = P\,\Sigma\,Q^T = A$.

(b) $A A^+ A = (P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T)(P\,\Sigma\,Q^T) = P\,\Sigma\,\Sigma^{-1}\,\Sigma\,Q^T = P\,\Sigma\,Q^T = A$.

(c) $A^+ A A^+ = (Q\,\Sigma^{-1} P^T)(P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T) = Q\,\Sigma^{-1}\,\Sigma\,\Sigma^{-1} P^T = Q\,\Sigma^{-1} P^T = A^+$. Or, you can use the fact that $(A^+)^+ = A$.

(d) $(A A^+)^T = (Q\,\Sigma^{-1} P^T)^T (P\,\Sigma\,Q^T)^T = P(\Sigma^{-1})^T Q^T Q \Sigma^T P^T = P(\Sigma^{-1})^T \Sigma^T P^T = P P^T = P\,\Sigma^{-1} \Sigma\, P^T = (P\,\Sigma\,Q^T)(Q\,\Sigma^{-1} P^T) = A A^+$.

(e) This follows from part (d) since $(A^+)^+ = A$.

8.5.30. Suppose $\mathbf{b} \in \operatorname{rng} A$ and $\ker A = \{\mathbf{0}\}$. Prove that $\mathbf{x} = A^+\mathbf{b}$ is the unique solution to the linear system $A\,\mathbf{x} = \mathbf{b}$. What if $\ker A \neq \{\mathbf{0}\}$?

Solution: ■

Jordan

8.6. Incomplete Matrices and the Jordan Canonical Form.

8.6.1. For each of the following Jordan matrices, identify the Jordan blocks. Write down the eigenvalues, the eigenvectors, and the Jordan basis. Clearly identify the Jordan chains.

$$(a)\ \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},\quad (b)\ \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix},\quad (c)\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\quad (d)\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\quad (e)\ \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Solution:
(*a*) One 2×2 Jordan block; eigenvalue 2; eigenvector $\mathbf{e}_1$.
(*b*) Two 1×1 Jordan blocks; eigenvalues $-3, 6$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.
(*c*) One 1×1 and one 2×2 Jordan blocks; eigenvalue 1; eigenvectors $\mathbf{e}_1, \mathbf{e}_2$.
(*d*) One 3×3 Jordan block; eigenvalue 0; eigenvector $\mathbf{e}_1$.
(*e*) An 1×1, 2×2 and 1×1 Jordan blocks; eigenvalues $4, 3, 2$; eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_4$.

8.6.2. Write down all possible 4×4 Jordan matrices that only have an eigenvalue 2.

Solution:

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

8.6.3. Write down all possible 3×3 Jordan matrices that have eigenvalues 2 and 5 (and no others).

Solution:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix},\quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},\quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix},\quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix},\quad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix},\quad \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

8.6.4. Find Jordan bases and the Jordan canonical form for the following matrices:

$$(a)\ \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix},\qquad (b)\ \begin{pmatrix} -1 & -1 \\ 4 & -5 \end{pmatrix},\qquad (c)\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$
$$(d)\ \begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix},\qquad (e)\ \begin{pmatrix} -1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix},\qquad (f)\ \begin{pmatrix} 2 & -1 & 1 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Solution:
(*a*) Eigenvalue: 2. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ \frac{1}{3} \end{pmatrix}$. Jordan canonical form: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.
(*b*) Eigenvalue: -3. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$. Jordan canonical form: $\begin{pmatrix} -3 & 1 \\ 0 & -3 \end{pmatrix}$.
(*c*) Eigenvalue: 1. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Jordan canonical form: $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(d) Eigenvalue: -3. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Jordan canonical form: $\begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{pmatrix}$.

(e) Eigenvalues: $-2, 0$. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$.

Jordan canonical form: $\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(f) Eigenvalue: 2. Jordan basis: $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -1 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{pmatrix}$.

Jordan canonical form: $\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

8.6.5. Write down a formula for the inverse of a Jordan block matrix. *Hint*: Try some small examples first to help figure out the pattern.

Solution:

$$J_{\lambda,n}^{-1} = \begin{pmatrix} \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & -\lambda^{-4} & \ldots & -(-\lambda)^{n} \\ 0 & \lambda^{-1} & -\lambda^{-2} & \lambda^{-3} & \ldots & -(-\lambda)^{n-1} \\ 0 & 0 & \lambda^{-1} & -\lambda^{-2} & \ldots & -(-\lambda)^{n-2} \\ 0 & 0 & 0 & \lambda^{-1} & \ldots & -(-\lambda)^{n-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & \lambda^{-1} \end{pmatrix}.$$

8.6.6. *True or false*: If A is complete, every generalized eigenvector is an ordinary eigenvector.

Solution: True. All Jordan chains have length one, and so consist only of eigenvectors.

♢ 8.6.7. Suppose you know all eigenvalues of a matrix as well as their algebraic and geometric multiplicities. Can you determine the matrix's Jordan canonical form?

Solution: No in general. If an eigenvalue has multiplicity ≤ 3, then you can tell the size of its Jordan blocks by the number of linearly independent eigenvectors it has: if it has 3 linearly independent eigenvectors, then there are 3 1×1 Jordan blocks; if it has 2 linearly independent eigenvectors then there are Jordan blocks of sizes 1×1 and 2×2, while if it only has one linearly independent eigenvector, then it corresponds to a 3×3 Jordan block. But if the multiplicity of the eigenvalue is 4, and there are only 2 linearly independent eigenvectors, then it could have two 2×2 blocks, or a 1×1 and an 3×3 block. Distinguishing between the two cases is a difficult computational problem.

8.6.8. *True or false*: If $\mathbf{w}_1, \ldots, \mathbf{w}_j$ is a Jordan chain for a matrix A, so are the scalar multiples $c\mathbf{w}_1, \ldots, c\mathbf{w}_j$ for any $c \neq 0$.

Solution: True. If $\mathbf{z}_j = c\mathbf{w}_j$, then $A\mathbf{z}_j = cA\mathbf{w}_j = c\lambda\mathbf{w}_j + c\mathbf{w}_{j-1} = \lambda\mathbf{z}_j + \mathbf{z}_{j-1}$.

8.6.9. *True or false*: If A has Jordan canonical form J, then A^2 has Jordan canonical J^2.

Solution: False. Indeed, the square of a Jordan matrix is not necessarily a Jordan matrix, e.g.,
$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

♢ 8.6.10. (*a*) Give an example of a matrix A such that A^2 has an eigenvector that is *not* an eigenvector of A. (*b*) Show that every eigenvalue of A^2 is the square of an eigenvalue of A.

Solution:

(*a*) Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{e}_2$ is an eigenvector of $A^2 = \mathrm{O}$, but is not an eigenvector of A.

(*b*) Suppose $A = S\,J\,S^{-1}$ where J is the Jordan canonical form of A. Then $A^2 = S\,J^2 S^{-1}$. Now, even though J^2 is not necessarily a Jordan matrix, cf. Exercise 8.6.9, since J is upper triangular with the eigenvalues on the diagonal, J^2 is also upper triangular and its diagonal entries, which are its eigenvalues and the eigenvalues of A^2, are the squares of the diagonal entries of J. *Q.E.D.*

8.6.11. Let A and B be $n \times n$ matrices. According to Exercise 8.2.22, the matrix products $A\,B$ and $B\,A$ have the same eigenvalues. Do they have the same Jordan form?

Solution: No.
A simple example is $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so $A\,B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, whereas $B\,A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

♢ 8.6.12. Prove Lemma 8.49.

Solution: First, since $J_{\lambda,n}$ is upper triangular, its eigenvalues are on the diagonal, and hence λ is the only eigenvalue. Moreover, $\mathbf{v} = (\, v_1, v_2, \ldots, v_n \,)^T$ is an eigenvector if and only if $(J_{\lambda,n} - \lambda\,\mathrm{I})\mathbf{v} = (\, v_2, \ldots, v_n, 0 \,)^T = \mathbf{0}$. This requires $v_2 = \cdots = v_n = 0$, and hence $\mathbf{v}$ must be a scalar multiple of $\mathbf{e}_1$. *Q.E.D.*

♢ 8.6.13. (*a*) Prove that a Jordan block matrix $J_{0,n}$ with zero diagonal entries is nilpotent, as in Exercise 1.3.13. (*b*) Prove that a Jordan matrix is nilpotent if and only if all its diagonal entries are zero. (*c*) Prove that a matrix is nilpotent if and only if its Jordan canonical form is nilpotent. (*d*) Explain why a matrix is nilpotent if and only if its only eigenvalue is 0.

Solution:

(*a*) Observe that $J_{0,n}^k$ is the matrix with 1's along the $(k+1)^{\text{st}}$ upper diagonal, i.e., in positions $(i, k+i+1)$. Thus, for $k = n$, all entries are all 0, and so $J_{0,n}^n = \mathrm{O}$.

(*b*) Since a Jordan matrix is upper triangular, the diagonal entries of J^k are the k^{th} powers of diagonal entries of J, and hence $J^m = \mathrm{O}$ requires that all its diagonal entries are zero. Moreover, J^k is a block matrix whose blocks are the k^{th} powers of the original Jordan blocks, and so $J^m = \mathrm{O}$ where m is the maximal size Jordan block.

(*c*) If $A = S\,J\,S^{-1}$, then $A^k = S\,J^k S^{-1}$ and hence $A^k = \mathrm{O}$ if and only if $J^k = \mathrm{O}$.

(*d*) This follow from parts (*c–d*).

8.6.14. Let J be a Jordan matrix. (*a*) Prove that J^k is a complete matrix for some $k \geq 1$ if and only if either J is diagonal, or J is nilpotent with $J^k = \mathrm{O}$. (*b*) Suppose that A is an incomplete matrix such that A^k is complete for some $k \geq 2$. Prove that $A^k = \mathrm{O}$. (A simpler version of this problem appears in Exercise 8.3.8.)

Solution:

(*a*) Since J^k is upper triangular, Exercise 8.3.12 says it is complete if and only if it is a diagonal matrix, which is the case if and only if J is diagonal, or $J^k = \mathrm{O}$.

(b) Write $A = S\,J\,S^{-1}$ in Jordan canonical form. Then $A^k = S\,J^k\,S^{-1}$ is complete if and only if J^k is complete, so either J is diagonal, and so A is complete, or $J^k = \mathrm{O}$ and so $A^k = \mathrm{O}$.

♡ 8.6.15. *The Schur Decomposition*: In practical computations, there are severe numerical difficulties in constructing the Jordan canonical form, [**59**]. A simpler version, due to I. Schur, often suffices in applications. The result to be proved is that if A is any $n \times n$ matrix, then there exists a unitary matrix Q, as defined in Exercise 5.3.25, such that $Q^\dagger A Q = U$ is upper triangular. (a) Explain why A and U have the same eigenvalues. How are their eigenvectors related? (b) To establish the Schur Decomposition, we work by induction. First show that if Q_1 is an unitary matrix that has a unit eigenvector of A as its first column, then $Q_1^\dagger A Q_1 = \begin{pmatrix} \lambda & \mathbf{c} \\ \mathbf{0} & B \end{pmatrix}$ where λ is the eigenvalue, and B has size $(n-1)\times(n-1)$. Then use the induction hypothesis for B to complete the induction step. (c) Is the Schur Decomposition unique? (d) Show that if A has all real eigenvalues, then Q will be an orthogonal matrix. (e) Show that if A is symmetric, then U is diagonal. Which result do we recover? (f) Establish a Schur Decomposition for the following matrices:

(i) $\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, ($ii$) $\begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$, ($iii$) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$, ($iv$) $\begin{pmatrix} -1 & -1 & 4 \\ 1 & 3 & -2 \\ 1 & 1 & -1 \end{pmatrix}$, ($v$) $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$.

Solution:

(a) $\blacksquare$

♡ 8.6.16. The *Cayley-Hamilton Theorem*: Let $p_A(\lambda) = \det(A - \lambda\,\mathrm{I}\,)$ be the characteristic polynomial of A. (a) Prove that if D is a diagonal matrix, then† $p_D(D) = \mathrm{O}$. *Hint*: Leave $p_D(\lambda)$ in factored form. (b) Prove that if A is complete, then $p_A(A) = \mathrm{O}$. (c) Prove that if J is a Jordan block, then $p_J(J) = \mathrm{O}$. (d) Prove that this also holds if J is a Jordan matrix. (e) Prove that any matrix satisfies its own characteristic equation: $p_A(A) = \mathrm{O}$.

Solution:

(a) If $D = \operatorname{diag}(d_1, \ldots, d_n)$, then $p_D(\lambda) = \prod_{i=1}^{n} (\lambda - d_i)$. Now $D - d_i\,\mathrm{I}$ is a diagonal matrix with 0 in its i^{th} diagonal position. The entries of the product $p_D(D) = \prod_{i=1}^{n} (D - d_i\,\mathrm{I}\,)$ of diagonal matrices is the product of the individual diagonal entries, but each such product has at least one zero, and so the result is a diagonal matrix with all 0 diagonal entries, i.e., the zero matrix.

(b) First, according to Exercise 8.2.31, similar matrices have the same characteristic polynomials, and so if $A = S\,D\,S^{-1}$ then $p_A(\lambda) = p_D(\lambda)$. On the other hand, if $p(\lambda)$ is any polynomial, then $p(S\,D\,S^{-1}) = S^{-1} p(D)\,S$. Therefore, if A is complete, we can diagonalize $A = S\,D\,S^{-1}$, and so, by part (a) and the preceding two facts,
$$p_A(A) = p_A(S\,D\,S^{-1}) = S^{-1} p_A(D)\,S = S^{-1} p_D(D)\,S = \mathrm{O}.$$

(c) The characteristic polynomial of the upper triangular Jordan block matrix $J = J_{\mu,n}$ with eigenvalue μ is $p_J(\lambda) = (\lambda - \mu)^n$. Thus, $p_J(J) = (J - \mu\,\mathrm{I}\,)^n = J_{0,n}^n = \mathrm{O}$ by Exercise 8.6.13.

(d) The determinant of a (Jordan) block matrix is the product of the determinants of the

† See Exercise 1.2.33 for the basics of matrix polynomials.

individual blocks. Moreover, by part (c), substituting J into the product of the characteristic polynomials for its Jordan blocks gives zero in each block, and so the product matrix vanishes.

(e) Same argument as in part (b), using the fact that a matrix and its Jordan canonical form have the same characteristic polynomial.

$\diamondsuit$ 8.6.17. Prove that the n vectors constructed in the proof of Theorem 8.46 are linearly independent and hence form a Jordan basis. *Hint*: Suppose that some linear combination vanishes. Apply B to the equation, and then use the fact that we started with a Jordan basis for $W = \operatorname{rng} B$.

Solution: Suppose some linear combination vanishes:

$$c_1 \mathbf{w}_1 + \ \cdots \ c_{n-r+k} \mathbf{w}_{n-r+k} + d_1 \mathbf{z}_1 + \ \cdots \ d_{r-k} \mathbf{z}_{r-k} = \mathbf{0}.$$

We multiply by B, to find $c_1 B \mathbf{w}_1 + \ \cdots \ c_{n-r+k} B \mathbf{w}_{n-r+k} = \mathbf{0}$ since the $\mathbf{z}_i \in \ker B$. ■