Solutions — Chapter 9

dode 9.1. Basic Solution Techniques.

9.1.1. Choose one or more of the following differential equations, and then: (a) Solve the equation directly. (b) Write down its phase plane equivalent, and the general solution to the phase plane system. (c) Plot at least four representative trajectories to illustrate the phase portrait. (d) Choose two trajectories in your phase portrait and graph the corresponding solution curves $u(t)$. Explain in your own words how the orbit and the solution graph are related. (i) $\ddot{u} + 4u = 0$, (ii) $\ddot{u} - 4u = 0$, (iii) $\ddot{u} + 2\dot{u} + u = 0$, ($iv$) $\ddot{u} + 4\dot{u} + 3u = 0$, ($v$) $\ddot{u} - 2\dot{u} + 10u = 0$.

Solution:

(i) (a) $u(t) = c_1 \cos 2t + c_2 \sin 2t$. ($b$) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \mathbf{u}$. ($c$) $\mathbf{u}(t) = \begin{pmatrix} c_1 \cos 2t + c_2 \sin 2t \\ -2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

(ii) (a) $u(t) = c_1 e^{-2t} + c_2 e^{2t}$. ($b$) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix} \mathbf{u}$. ($c$) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-2t} + c_2 e^{2t} \\ -2c_1 e^{-2t} + 2c_2 e^{2t} \end{pmatrix}$.

(iii) (a) $u(t) = c_1 e^{-t} + c_2 t e^{-t}$. ($b$) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{u}$. ($c$) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ (c_2 - c_1) e^{-t} - c_2 t e^{-t} \end{pmatrix}$.

(iv) (a) $u(t) = c_1 e^{-t} + c_2 e^{-3t}$. ($b$) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix} \mathbf{u}$. ($c$) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{-3t} \\ -c_1 e^{-t} - 3c_2 e^{-3t} \end{pmatrix}$.

(v) (a) $u(t) = c_1 e^{t} \cos 3t + c_2 e^{t} \sin 3t$. ($b$) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 \\ -10 & 2 \end{pmatrix} \mathbf{u}$.

(c) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} \cos 3t + c_2 e^{-t} \sin 3t \\ -(c_1 + 3c_2) e^{-t} \cos 3t + (3c_1 - c_2) e^{-t} \sin 3t \end{pmatrix}$.

9.1.2. (a) Convert the third order equation $\dfrac{d^3u}{dt^3} + 3\dfrac{d^2u}{dt^2} + 4\dfrac{du}{dt} + 12u = 0$ into a first order system in three variables by setting $u_1 = u, u_2 = \dot{u}, u_3 = \ddot{u}$. ($b$) Solve the equation directly, and then use this to write down the general solution to your first order system. (c) What is the dimension of the solution space?

Solution:

(a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -4 & -3 \end{pmatrix} \mathbf{u}$.

(b) $u(t) = c_1 e^{-3t} + c_2 \cos 2t + c_3 \sin 2t$, $\quad \mathbf{u}(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 \cos 2t + c_3 \sin 2t \\ -3c_1 e^{-3t} - 2c_2 \sin 2t + 2c_3 \cos 2t \\ 9c_1 e^{-3t} - 4c_2 \cos 2t - 4c_3 \sin 2t \end{pmatrix}$,

(c) dimension $= 3$.

9.1.3. Convert the second order coupled system of ordinary differential equations $\ddot{u} = a\dot{u} + b\dot{v} + cu + dv$, $\ddot{v} = p\dot{u} + q\dot{v} + ru + sv$, into a first order system involving four variables.

Solution: Set $u_1 = u, u_2 = \dot u, u_3 = v, u_4 = \dot v$. Then $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ c & a & d & b \\ 0 & 0 & 0 & 1 \\ r & p & s & q \end{pmatrix} \mathbf{u}.$

9.1.4. *True or false*: The phase plane trajectories (9.9) for $(c_1, c_2)^T \neq \mathbf{0}$ are hyperbolas.

Solution: False — there is no quadratic equation in u_1, u_2 that they satisfy because $u_1^2, u_1 u_2, u_2^2$, u_1, u_2 and 1 are linearly independent functions.

♢ 9.1.5. (*a*) Show that if $\mathbf{u}(t)$ solves $\dot{\mathbf{u}} = A\mathbf{u}$, then its *time reversal*, defined as $\mathbf{v}(t) = \mathbf{u}(-t)$, solves $\dot{\mathbf{v}} = B\mathbf{v}$, where $B = -A$. (*b*) Explain why the two systems have the same phase portraits, but the direction of motion along the trajectories is reversed. (*c*) Apply time reversal to the system(s) you derived in Exercise 9.1.1. (*d*) What is the effect of time reversal on the original second order equation?

Solution:

(*a*) Use the chain rule to compute $\dfrac{d\mathbf{v}}{dt} = -\dfrac{d\mathbf{u}}{dt}(-t) = -A\mathbf{u}(-t) = -A\mathbf{v}$.

(*b*) Since $\mathbf{v}(t) = \mathbf{u}(-t)$ parametrizes the same curve as $\mathbf{u}(t)$, but in the reverse direction.

(*c*)

(*i*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}\mathbf{u}$; solution: $\mathbf{u}(t) = \begin{pmatrix} c_1 \cos 2t - c_2 \sin 2t \\ 2c_1 \sin 2t + 2c_2 \cos 2t \end{pmatrix}$.

(*ii*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ -4 & 0 \end{pmatrix}\mathbf{u}$; solution: $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-2t} \\ -2c_1 e^{2t} + 2c_2 e^{-2t} \end{pmatrix}$.

(*iii*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\mathbf{u}$; solution: $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{t} - c_2 t e^{t} \\ (c_2 - c_1) e^{t} + c_2 t e^{t} \end{pmatrix}$.

(*iv*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 3 & 4 \end{pmatrix}\mathbf{u}$; solution: $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{t} + c_2 e^{3t} \\ -c_1 e^{t} - 3c_2 e^{3t} \end{pmatrix}$.

(*v*) $\dfrac{d\mathbf{v}}{dt} = \begin{pmatrix} 0 & -1 \\ 10 & -2 \end{pmatrix}\mathbf{u}$; solution: $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{t} \cos 3t - c_2 e^{t} \sin 3t \\ -(c_1 + 3c_2) e^{t} \cos 3t - (3c_1 - c_2) e^{t} \sin 3t \end{pmatrix}$.

(*vi*) Time reversal changes $u_1(t) = u(t)$ into $v_1(t) = u_1(-t) = u(-t)$ and $u_2(t) = \dot u(t)$ into $v_2(t) = u_2(-t) = \dot u(-t) = -\dot v(t)$. The net effect is to change the sign of the coefficient of the first derivative term, so $\dfrac{d^2u}{dt^2} + a\dfrac{du}{dt} + bu = 0$ becomes $\dfrac{d^2v}{dt^2} - a\dfrac{dv}{dt} + bv = 0$.

9.1.6. (*a*) Show that if $\mathbf{u}(t)$ solves $\dot{\mathbf{u}} = A\mathbf{u}$, then $\mathbf{v}(t) = \mathbf{u}(2t)$ solves $\dot{\mathbf{v}} = B\mathbf{v}$, where $B = 2A$. (*b*) How are the solution trajectories of the two systems related?

Solution: (*a*) Use the chain rule to compute $\dfrac{d\mathbf{v}}{dt} = 2\dfrac{d\mathbf{u}}{dt}(2t) = 2A\mathbf{u}(2t) = 2A\mathbf{v}(t)$, and so the coefficient matrix is multiplied by 2. (*b*) The solution trajectories are the same, but the solution moves twice as fast (in the same direction) along them.

9.1.7. *True or false*: Each solution to a phase plane system moves at a constant speed along its trajectory.

Solution: False. If $\dot{\mathbf{u}} = A\mathbf{u}$ then the speed at the point $\mathbf{u}(t)$ on the trajectory is $\| A\mathbf{u}(t)) \|$. So the speed is constant only if $\| A\mathbf{u}(t)) \|$ is constant. (Later this will be shown to correspond to A being a skew-symmetric matrix.)

♠ 9.1.8. Use a three-dimensional graphics package to plot solution curves $(t, u_1(t), u_2(t))^T$ of the phase plane systems in Exercise 9.1.1. Discuss their shape and explain how they are related the phase plane trajectories.

Solution: ■ The solution curves project to the phase plane trajectories.

♡ 9.1.9. A first order linear system $\dot u = a u + b v$, $\dot v = c u + d v$, can be converted into a single second order differential equation by the following device. Assuming $b \neq 0$, solve the system for v and $\dot v$ in terms of u and $\dot u$. Then differentiate your equation for v with respect to t, and eliminate $\dot v$ from the resulting pair of equations. The result is a second order ordinary differential equation for $u(t)$. (*a*) Write out the second order equation in terms of the coefficients a, b, c, d of the first order system. (*b*) Show that there is a one-to-one correspondence between solutions of the system and solutions of the scalar differential equation. (*c*) Use this method to solve the following linear systems, and sketch the resulting phase portraits. (*i*) $\dot u = v, \dot v = -u$, (*ii*) $\dot u = 2u + 5v, \dot v = -u$, (*iii*) $\dot u = 4u - v, \dot v = 6u - 3v$, (*iv*) $\dot u = u + v, \dot v = u - v$, (*v*) $\dot u = v, \dot v = 0$. (*d*) Show how to obtain a second order equation satisfied by $v(t)$ by an analogous device. Are the second order equations for u and for v the same? (*e*) Discuss how you might proceed if $b = 0$.

Solution:

(a) For $b \neq 0$, we have

$$v = \frac{1}{b}\,\dot u - \frac{a}{b}\,u, \qquad \dot v = \frac{bc - ad}{b}\,u + \frac{d}{b}\,\dot u. \tag{$*$}$$

Differentiating the first equation yields

$$\frac{dv}{dt} = \frac{1}{b}\,\ddot u - \frac{a}{b}\,\dot u.$$

Equating this to the right hand side of the second equation yields leading to the second order differential equation

$$\ddot u - (a+d)\dot u + (ad - bc)u = 0. \tag{$**$}$$

(*b*) If $u(t)$ solves $(**)$, then defining $v(t)$ by the first equation in $(*)$ yields a solution to the first order system. Vice versa, the first component of any solution $u(t), v(t)$ to the system gives a solution $u(t)$ to the second order equation.

(*c*)

(*i*) $\ddot u + u = 0$, hence $u(t) = c_1 \cos t + c_2 \sin t$, $v(t) = -c_1 \sin t + c_2 \cos t$.

(*ii*) $\ddot u - 2\dot u + 5u = 0$, hence

$$u(t) = c_1 e^{t} \cos 2t + c_2 e^{t} \sin 2t, \quad v(t) = (c_1 + 2c_2)\, e^{t} \cos 2t + (-2c_1 + c_2)\, e^{t} \sin 2t.$$

(*iii*) $\ddot u - \dot u - 6u = 0$, hence $u(t) = c_1 e^{3t} + c_2 e^{-2t}$, $v(t) = c_1 e^{2t} + 6c_2 e^{-2t}$.

(*iv*) $\ddot u - 2u = 0$, hence $u(t) = c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t}$, $v(t) = (\sqrt{2}-1)c_1 e^{\sqrt{2}\,t} - (\sqrt{2}+1)c_2 e^{-\sqrt{2}\,t}$.

(*v*) $\ddot u = 0$, hence $u(t) = c_1 t + c_2$, $v(t) = c_1$.

(*d*) For $c \neq 0$ we can solve for $u = \dfrac{1}{d}\,\dot v - \dfrac{c}{d}\,v$, $\dot u = \dfrac{ad - bc}{d}\,u + \dfrac{b}{d}\,\dot v$, leading to the same second order equation for v, namely, $\ddot v - (a+d)\dot v + (ad - bc)v = 0$.

(*e*) If $b = 0$ then u solves a first order linear equation; once we solve the equation, we can then recover v by solving an inhomogeneous first order equation. Note that u continues to solve the same second order equation, but is no longer the most general solution and so the one-to-one correspondence between solutions breaks down.

9.1.10. Find the solution to the system of differential equations $\dfrac{du}{dt} = 3u + 4v$, $\dfrac{dv}{dt} = 4u - 3v$, with initial conditions $u(0) = 3$ and $v(0) = -2$.

Solution: $u(t) = \frac{7}{5}\, e^{-5t} + \frac{8}{5}\, e^{5t}, v(t) = -\frac{14}{5}\, e^{-5t} + \frac{4}{5}\, e^{5t}$.

9.1.11. Find the general real solution to the following systems of differential equations:

$$\text{(a)}\ \begin{aligned}\dot u_1 &= u_1 + 9u_2,\\ \dot u_2 &= u_1 + 3u_2.\end{aligned}\qquad \text{(b)}\ \begin{aligned}\dot x_1 &= 4x_1 + 3x_2,\\ \dot x_2 &= 3x_1 - 4x_2.\end{aligned}\qquad \text{(c)}\ \begin{aligned}\dot y_1 &= y_1 - y_2,\\ \dot y_2 &= 2y_1 + 3y_2.\end{aligned}$$

$$\text{(d)}\ \begin{aligned}\dot y_1 &= y_2,\\ \dot y_2 &= 3y_1 + 2y_3,\\ \dot y_3 &= -y_2.\end{aligned}\qquad \text{(e)}\ \begin{aligned}\dot x_1 &= 3x_1 - 8x_2 + 2x_3,\\ \dot x_2 &= -x_1 + 2x_2 + 2x_3,\\ \dot x_3 &= x_1 - 4x_2 + 2x_3.\end{aligned}$$

Solution:

(a) $u_1(t) = (\sqrt{10}-1)c_1 e^{(2+\sqrt{10})t} - (\sqrt{10}+1)c_2 e^{(2-\sqrt{10})t}$, $u_2(t) = c_1 e^{(2+\sqrt{10})t} + c_2 e^{(2-\sqrt{10})t}$.

(b) $x_1(t) = -c_1 e^{-5t} + 3c_2 e^{5t}$, $x_2(t) = 3c_1 e^{-5t} + c_2 e^{5t}$.

(c) $y_1(t) = e^{2t}\big[c_1 \cos t - (c_1 + c_2)\sin t\big]$, $y_2(t) = e^{2t}\big[c_2 \cos t + (2c_1 + c_2)\sin t\big]$.

(d) $y_1(t) = -c_1 e^{-t} - c_2 e^{t} - \frac{2}{3}c_3$, $y_2(t) = c_1 e^{-t} - c_2 e^{t}$, $y_3(t) = c_1 e^{-t} + c_2 e^{t} + c_3$.;

(e) $x_1(t) = 3c_1 e^{t} + 2c_2 e^{2t} + 2c_3 e^{4t}$, $x_2(t) = c_1 e^{t} + \frac{1}{2}c_2 e^{2t}$, $x_3(t) = c_1 e^{t} + c_2 e^{2t} + c_3 e^{4t}$.

9.1.12. Solve the following initial value problems: (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 2\\ 2 & 0\end{pmatrix}\mathbf{u}$, $\mathbf{u}(1) = \begin{pmatrix}1\\0\end{pmatrix}$,

(b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & -2\\ -2 & 1\end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix}-2\\4\end{pmatrix}$, (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & 2\\ -1 & 1\end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix}1\\0\end{pmatrix}$,

(d) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 3 & -3\\ 2 & 2 & -7\\ 0 & 3 & -4\end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$, (e) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 1 & -6\\ -1 & 0 & 4\\ 0 & -1 & -2\end{pmatrix}\mathbf{u}$, $\mathbf{u}(\pi) = \begin{pmatrix}2\\-1\\-1\end{pmatrix}$,

(f) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 2\\ 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\end{pmatrix}\mathbf{u}$, $\mathbf{u}(2) = \begin{pmatrix}1\\0\\0\\1\end{pmatrix}$, (g) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 1 & -1 & 0\\ -3 & -2 & 0 & 1\\ 0 & 0 & 1 & -2\\ 0 & 0 & 1 & -1\end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix}1\\-1\\2\\1\end{pmatrix}$.

Solution:

(a) $\mathbf{u}(t) = \big(\frac{1}{2}e^{2-2t} + \frac{1}{2}e^{-2+2t}, -\frac{1}{2}e^{2-2t} + \frac{1}{2}e^{-2+2t}\big)^T$;

(b) $\mathbf{u}(t) = \big(e^{-t} - 3e^{3t}, e^{-t} + 3e^{3t}\big)^T$;

(c) $\mathbf{u}(t) = \Big(e^{t}\cos\sqrt{2}\,t, -\frac{1}{\sqrt{2}}\,e^{t}\sin\sqrt{2}\,t\Big)^T$;

(d) $\mathbf{u}(t) = \Big(e^{-t}\big(2 - \cos\sqrt{6}\,t\big), e^{-t}\big(1 - \cos\sqrt{6}\,t + \sqrt{\tfrac{2}{3}}\sin\sqrt{6}\,t\big), e^{-t}\big(1 - \cos\sqrt{6}\,t\big)\Big)^T$;

(e) $\mathbf{u}(t) = (-4 - 6\cos t - 9\sin t, 2 + 3\cos t + 6\sin t, -1 - 3\sin t)^T$;

(f) $\mathbf{u}(t) = \big(\frac{1}{2}e^{2-t} + \frac{1}{2}e^{-2+t}, -\frac{1}{2}e^{4-2t} + \frac{1}{2}e^{-4+2t}, -\frac{1}{2}e^{2-t} + \frac{1}{2}e^{-2+t}, \frac{1}{2}e^{4-2t} + \frac{1}{2}e^{-4+2t}\big)^T$;

(g) $\mathbf{u}(t) = \big(-\frac{1}{2}e^{-t} + \frac{3}{2}\cos t - \frac{3}{2}\sin t, \frac{3}{2}e^{-t} - \frac{5}{2}\cos t + \frac{3}{2}\sin t, 2\cos t, \cos t + \sin t\big)^T$.

9.1.13. (a) Find the solution to the system $\dfrac{dx}{dt} = -x + y$, $\dfrac{dy}{dt} = -x - y$, that has initial conditions $x(0) = 1$, $y(0) = 0$. (b) Sketch a phase portrait of the system that shows several typical solution trajectories, including the solution you found in part (a). Clearly indicate the direction of increasing t on your curves.

Solution: $x(t) = e^{-t}\cos t, y(t) = -e^{-t}\sin t$; ■ the orbits spiral in a clockwise dirfection, approaching the origin exponentially fast as $t \to \infty$.

9.1.14. A planar steady state fluid flow has velocity vector $\mathbf{v} = (2x - 3y, x - y)^T$. The motion of the fluid is described by the differential equation $\dfrac{d\mathbf{x}}{dt} = \mathbf{v}$. A floating object starts out at the point $(1,1)^T$. Find its position after 1 time unit.

Solution: $x(t) = e^{t/2}\left(\cos\frac{\sqrt{3}}{2}\,t - \sqrt{3}\,\sin\frac{\sqrt{3}}{2}\,t\right), y(t) = e^{t/2}\left(\cos\frac{\sqrt{3}}{2}\,t - \frac{1}{\sqrt{3}}\,\sin\frac{\sqrt{3}}{2}\,t\right)$, so $(\,x(1), y(1)\,)^T = (\,-1.10719, .343028\,)^T$.

9.1.15. A steady state fluid flow has velocity vector $\mathbf{v} = (\,-2y, 2x, z\,)^T$. Describe the motion of the fluid particles as governed by the differential equation $\dfrac{d\mathbf{x}}{dt} = \mathbf{v}$.

Solution: $\mathbf{x}(t) = \left(c_1\cos 2t - c_2\sin 2t, c_1\sin 2t + c_2\cos 2t, c_3\,e^{-t}\right)^T$. The fluid particles circle counterclockwise around the z axis at angular velocity 2, while spiraling closer and closer, at an exponential rate, to the xy plane. In the limit, they mover around the unit cirle in the xy plane at constant speed 2.

9.1.16. Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -6 & 1 \\ 1 & -6 \end{pmatrix}\mathbf{u},\ \ \mathbf{u}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Explain how orthogonality can help.

Solution: The coefficient matrix has eigenvalues $\lambda_1 = -5,\ \lambda_2 = -7$, and, since the coefficient matrix is symmetric, orthogonal eigenvectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The general solution is

$$\mathbf{u}(t) = c_1 e^{-5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-7t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

For the initial conditions

$$\mathbf{u}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

we can use orthogonality to find

$$c_1 = \frac{\langle\,\mathbf{u}(0)\,,\mathbf{v}_1\,\rangle}{\|\,\mathbf{v}_1\,\|^2} = \tfrac{2}{3}, \qquad\qquad c_2 = \frac{\langle\,\mathbf{u}(0)\,,\mathbf{v}_2\,\rangle}{\|\,\mathbf{v}_2\,\|^2} = \tfrac{1}{2}.$$

Therefore, the solution is

$$\mathbf{u}(t) = \tfrac{3}{2}\,e^{-5t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \tfrac{1}{2}\,e^{-7t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

9.1.17. (*a*) Find the eigenvalues and eigenvectors of $K = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. (*b*) Verify that the eigenvectors are mutually orthogonal. (*c*) Based on part (*a*), is K positive definite, positive semi-definite or indefinite? (*d*) Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = K\,\mathbf{u},\ \ \mathbf{u}(0) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, using orthogonality to simplify the computations.

Solution:

(*a*) eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 3$, eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

(*b*) The matrix is positive semi-definite since it has one zero eigenvalue and the rest are positive.

(*c*) The general solution is

$$\mathbf{u}(t) = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\,e^{t}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3\,e^{3t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

For the initial conditions

$$\mathbf{u}(0) = c_1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \mathbf{u}_0,$$

we can use orthogonality to find

$$c_1 = \frac{\langle \mathbf{u}_0 , \mathbf{v}_1 \rangle}{\| \mathbf{v}_1 \|^2} = \tfrac{2}{3}, \qquad c_2 = \frac{\langle \mathbf{u}_0 , \mathbf{v}_2 \rangle}{\| \mathbf{v}_2 \|^2} = 1, \qquad c_3 = \frac{\langle \mathbf{u}_0 , \mathbf{v}_3 \rangle}{\| \mathbf{v}_3 \|^2} = -\tfrac{2}{3}.$$

Therefore, the solution is $\mathbf{u}(t) = \left(\frac{2}{3} + e^t - \frac{2}{3}\, e^{3t}, \frac{2}{3} + \frac{4}{3}\, e^{3t}, \frac{2}{3} - e^t - \frac{2}{3}\, e^{3t} \right)^T.$

9.1.18. Demonstrate that one can also solve the initial value problem in Example 9.8 by writing the solution as a complex linear combination of the complex eigensolutions, and then using the initial conditions to specify the coefficients.

Solution: The general complex solution to the system is

$$\mathbf{u}(t) = c_1\, e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + c_2\, e^{(1+2\,\mathrm{i})\,t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + c_3\, e^{(1-2i)\,t} \begin{pmatrix} 1 \\ -\mathrm{i} \\ 1 \end{pmatrix}.$$

Substituting into the initial conditions,

$$\mathbf{u}(0) = \begin{pmatrix} -c_1 + c_2 + c_3 \\ c_1 + \mathrm{i}\, c_2 - \mathrm{i}\, c_3 \\ c_1 + + c_2 + c_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \qquad \text{and so} \qquad \begin{aligned} c_1 &= -2, \\ c_2 &= -\tfrac{1}{2}\,\mathrm{i}, \\ c_3 &= \tfrac{1}{2}\,\mathrm{i}. \end{aligned}$$

Thus, we obtain the same solution:

$$\mathbf{u}(t) = -2\, e^{-t} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \tfrac{1}{2}\,\mathrm{i}\, e^{(1+2\,\mathrm{i})\,t} \begin{pmatrix} 1 \\ \mathrm{i} \\ 1 \end{pmatrix} + \tfrac{1}{2}\,\mathrm{i}\, e^{(1-2i)\,t} \begin{pmatrix} 1 \\ -\mathrm{i} \\ 1 \end{pmatrix} = \begin{pmatrix} 2\, e^{-t} + e^t \sin 2t \\ -2\, e^{-t} + e^t \cos 2t \\ -2\, e^{-t} + e^t \sin 2t \end{pmatrix}.$$

9.1.19. Determine whether the following vector-valued functions are linearly dependent or linearly independent:

(a) $\begin{pmatrix} 1 \\ t \end{pmatrix}, \begin{pmatrix} -t \\ 1 \end{pmatrix}$, (b) $\begin{pmatrix} 1+t \\ t \end{pmatrix}, \begin{pmatrix} 1-t^2 \\ t-t^2 \end{pmatrix}$, (c) $\begin{pmatrix} 1 \\ t \end{pmatrix}, \begin{pmatrix} t \\ 2 \end{pmatrix}, \begin{pmatrix} -t \\ t \end{pmatrix}$, (d) $\begin{pmatrix} e^{-t} \\ -e^t \end{pmatrix}, \begin{pmatrix} -e^{-t} \\ e^t \end{pmatrix}$,

(e) $\begin{pmatrix} e^{2t} \cos 3t \\ -e^{2t} \sin 3t \end{pmatrix}, \begin{pmatrix} e^{2t} \sin 3t \\ e^{2t} \cos 3t \end{pmatrix}$, (f) $\begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}, \begin{pmatrix} \sin 3t \\ \cos 3t \end{pmatrix}$, (g) $\begin{pmatrix} 1 \\ t \\ 1-t \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1+3t \\ 2-3t \end{pmatrix}$,

(h) $\begin{pmatrix} e^t \\ -e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \\ -e^t \end{pmatrix}, \begin{pmatrix} -e^t \\ e^t \\ e^t \end{pmatrix}$, (i) $\begin{pmatrix} e^t \\ t e^t \\ t^2 e^t \end{pmatrix}, \begin{pmatrix} t^2 e^t \\ e^t \\ t e^t \end{pmatrix}, \begin{pmatrix} t e^t \\ t^2 e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}$.

Solution: Only (*d*), (*g*) are linearly dependent.

9.1.20. Let A be a constant matrix. Suppose $\mathbf{u}(t)$ solves the initial value problem $\dot{\mathbf{u}} = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{b}$. Prove that the solution to the initial value problem $\dot{\mathbf{u}} = A\mathbf{u}$, $\mathbf{u}(t_0) = \mathbf{b}$, is equal to $\widetilde{\mathbf{u}}(t) = \mathbf{u}(t - t_0)$. How are the solution trajectories related?

Solution: Using the chain rule, $\dfrac{d}{dt}\,\widetilde{\mathbf{u}}(t) = \dfrac{d\mathbf{u}}{dt}\,(t - t_0) = A\,\mathbf{u}(t - t_0) = A\,\widetilde{\mathbf{u}}$, and hence $\widetilde{\mathbf{u}}(t)$ solves the differential equation. Moreover, $\widetilde{\mathbf{u}}(t_0) = \mathbf{u}(0) = \mathbf{b}$ has the correct initial conditions. The trajectories are the same curves, but $\widetilde{\mathbf{u}}(t)$ is always ahead of $\mathbf{u}(t)$ by an amount t_0.

9.1.21. Prove that the general solution to a linear system $\dot{\mathbf{u}} = \Lambda\,\mathbf{u}$ with diagonal coefficient matrix $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is given by $\mathbf{u}(t) = \left(c_1\, e^{\lambda_1 t}, \ldots, c_n\, e^{\lambda_n t} \right)^T$.

Solution:

$$\frac{d\mathbf{u}}{dt} = \left(\lambda_1\, c_1\, e^{\lambda_1 t}, \ldots, \lambda_n\, c_n\, e^{\lambda_n t} \right)^T = \Lambda\,\mathbf{u}.$$

9.1.22. Show that if $\mathbf{u}(t)$ is a solution to $\dot{\mathbf{u}} = A\,\mathbf{u}$, and S is a constant, nonsingular matrix of the same size as A, then $\mathbf{v}(t) = S\,\mathbf{u}(t)$ solves the linear system $\dot{\mathbf{v}} = B\,\mathbf{v}$, where $B = S\,A\,S^{-1}$ is similar to A.

Solution: $\dfrac{d\mathbf{v}}{dt} = S\,\dfrac{d\mathbf{u}}{dt} = S\,A\,\mathbf{u} = S\,A\,S^{-1}\mathbf{v} = B\,\mathbf{v}$.

♢ 9.1.23. (*i*) Combine Exercises 9.1.21–22 to show that if $A = S\,\Lambda\,S^{-1}$ is diagonalizable, then the solution to $\dot{\mathbf{u}} = A\,\mathbf{u}$ can be written as $\mathbf{u}(t) = S\,\big(c_1\,e^{\lambda_1 t}, \ldots, c_n\,e^{\lambda_n t}\big)^T$, where $\lambda_1, \ldots, \lambda_n$ are its eigenvalues and $S = (\,\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n\,)$ is the corresponding matrix of eigenvectors. (*ii*) Write the general solution to the systems in Exercise 9.1.12 in this form.

Solution:
(*i*) This is an immediate consequence of the preceding two exercises.
(*ii*) (a) $\mathbf{u}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{-2t} \\ c_2\,e^{2t} \end{pmatrix}$; (b) $\mathbf{u}(t) = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{3t} \\ c_2\,e^{-t} \end{pmatrix}$;

(c) $\mathbf{u}(t) = \begin{pmatrix} -\sqrt{2}\,\mathrm{i} & \sqrt{2}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{(1+\mathrm{i}\sqrt{2})t} \\ c_2\,e^{(1-\mathrm{i}\sqrt{2})t} \end{pmatrix}$;

(d) $\mathbf{u}(t) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1+\mathrm{i}\sqrt{\frac{2}{3}} & 1-\mathrm{i}\sqrt{\frac{2}{3}} \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{-t} \\ c_2\,e^{(-1+\mathrm{i}\sqrt{6})t} \\ c_3\,e^{(-1-\mathrm{i}\sqrt{6})t} \end{pmatrix}$;

(e) $\mathbf{u}(t) = \begin{pmatrix} 4 & 3+2\,\mathrm{i} & 3-2\,\mathrm{i} \\ -2 & -2-\mathrm{i} & -2+\mathrm{i} \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2\,e^{\mathrm{i}t} \\ c_3\,e^{-\mathrm{i}t} \end{pmatrix}$; (f) $\mathbf{u}(t) = \begin{pmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1\,e^{-2t} \\ c_2\,e^{-t} \\ c_3\,e^{t} \\ c_4\,e^{2t} \end{pmatrix}$;

(g) $\mathbf{u}(t) = \begin{pmatrix} -1 & -1 & \frac{3}{2}\,\mathrm{i} & -\frac{3}{2}\,\mathrm{i} \\ 1 & 3 & -\frac{1}{2}-2\,\mathrm{i} & -\frac{1}{2}+2\,\mathrm{i} \\ 0 & 0 & 1+\mathrm{i} & 1-\mathrm{i} \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1\,e^{t} \\ c_2\,e^{-t} \\ c_3\,e^{\mathrm{i}t} \\ c_4\,e^{-\mathrm{i}t} \end{pmatrix}$.

9.1.24. Find the general solution to the linear system $\dfrac{d\mathbf{u}}{dt} = A\,\mathbf{u}$ for the following incomplete coefficient matrices: (a) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, (b) $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & -1 \\ 4 & -5 \end{pmatrix}$, (d) $\begin{pmatrix} 4 & -1 & -3 \\ -2 & 1 & 2 \\ 5 & -1 & -4 \end{pmatrix}$, (e) $\begin{pmatrix} -3 & 1 & 0 \\ 1 & -3 & -1 \\ 0 & 1 & -3 \end{pmatrix}$, (f) $\begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$, (g) $\begin{pmatrix} 3 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$, (h) $\begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$.

Solution: (a) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{2t} + c_2 t\,e^{2t} \\ c_2 e^{2t} \end{pmatrix}$, (b) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2\big(\frac{1}{3}+t\big)\,e^{-t} \\ 3\,c_1 e^{-t} + 3\,c_2 t\,e^{-t} \end{pmatrix}$,

(c) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-3t} + c_2\big(\frac{1}{2}+t\big)\,e^{-3t} \\ 2\,c_1 e^{-3t} + 2\,c_2 t\,e^{-3t} \end{pmatrix}$, (d) $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{t} + c_3 t\,e^{t} \\ -c_1 e^{-t} - c_3 e^{t} \\ 2\,c_1 e^{-t} + c_2 e^{t} + c_3 t\,e^{t} \end{pmatrix}$,

$(e)\ \ \mathbf{u}(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 t\, e^{-3t} + c_3 \left(1 + \frac{1}{2} t^2\right) e^{-3t} \\ c_2\, e^{-3t} + c_3 t\, e^{-3t} \\ c_1 e^{-3t} + c_2 t e^{-3t} + \frac{1}{2} c_3 t^2 e^{-3t} \end{pmatrix}$, $(f)\ \begin{pmatrix} c_1 e^{3t} + c_2 t e^{3t} - \frac{1}{4} c_3 e^{-t} - \frac{1}{4} c_4 (t+1) e^{-t} \\ c_3 e^{-t} + c_4 t e^{-t} \\ c_2 e^{3t} - \frac{1}{4} c_4\, e^{-t} \\ c_4\, e^{-t} \end{pmatrix}$,

$(g)\ \begin{pmatrix} c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \cos t \\ -c_1 \sin t + c_2 \cos t - c_3 t \sin t + c_4 t \cos t \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{pmatrix}$.

9.1.25. Find a first order system of ordinary differential equations that has the indicated vector-valued function as a solution: $(a)\ \begin{pmatrix} e^t + e^{2t} \\ 2e^t \end{pmatrix}$, $(b)\ \begin{pmatrix} e^{-t} \cos 3t \\ -3e^{-t} \sin 3t \end{pmatrix}$, $(c)\ \begin{pmatrix} 1 \\ t-1 \end{pmatrix}$, $(d)\ \begin{pmatrix} \sin 2t - \cos 2t \\ \sin 2t + 3 \cos 2t \end{pmatrix}$, $(e)\ \begin{pmatrix} e^{2t} \\ e^{-3t} \\ e^{2t} - e^{-3t} \end{pmatrix}$, $(f)\ \begin{pmatrix} \sin t \\ \cos t \\ 1 \end{pmatrix}$, $(g)\ \begin{pmatrix} t \\ 1 - t^2 \\ 1 + t \end{pmatrix}$, $(h)\ \begin{pmatrix} e^t \sin t \\ 2 e^t \cos t \\ e^t \sin t \end{pmatrix}$.

Solution: (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \mathbf{u}$, $(b)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 1 \\ -9 & -1 \end{pmatrix} \mathbf{u}$, $(c)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{u}$,
$(d)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & 1 \\ -5 & -1 \end{pmatrix} \mathbf{u}$, $(e)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 2 & 3 & 0 \end{pmatrix} \mathbf{u}$, $(f)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{u}$,
$(g)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \mathbf{u}$, $(h)\ \dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ -1 & 1 & -1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \mathbf{u}$.

9.1.26. Which sets of functions in Exercise 9.1.19 can be solutions to a common first order, homogeneous, constant coefficient linear system of ordinary differential equations? If so, find a system they satisfy; if not, explain why not.

Solution: (a) No, since neither $\dfrac{d\mathbf{u}_i}{dt}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. Or note that the trajectories described by the solutions cross, violating uniqueness. (b) No, since polynomial solutions a two-dimensional system can be at most first order in t. (c) No, since a two-dimensional system has at most 2 linearly independent solutions. (d) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{u}$. (e) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \mathbf{u}$. (f) No, since neither $\dfrac{d\mathbf{u}_i}{dt}$ is a linear combination of $\mathbf{u}_1, \mathbf{u}_2$. Or note that both solutions have the unit circle as their trajectory, but traverse it in opposite directions, violating uniqueness. (g) Yes: $\dot{\mathbf{u}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \mathbf{u}$. (h) Yes: $\dot{\mathbf{u}} = \mathbf{u}$. (i) No, since a three-dimensional system has at most 3 linearly independent solutions.

9.1.27. Solve the third order equation $\dfrac{d^3 u}{dt^3} + 3 \dfrac{d^2 u}{dt^2} + 4 \dfrac{du}{dt} + 12\, u = 0$ by converting it into a first order system. Compare your answer with what you found in Exercise 9.1.2.

Solution: Setting $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \\ \ddot{u}(t) \end{pmatrix}$, the first order system is $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -4 & -3 \end{pmatrix} \mathbf{u}$. The eigenvalues of the coefficient matrix are $-3, \pm 2\,\mathrm{i}$ with eigenvectors $\begin{pmatrix} 1 \\ -3 \\ 9 \end{pmatrix}, \begin{pmatrix} 1 \\ \pm 2\,\mathrm{i} \\ -4 \end{pmatrix}$ and the resulting solution is $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-3t} + c_2 \cos 2t + c_3 \sin 2t \\ -3c_1 e^{-3t} - 2c_2 \sin 2t + 2c_3 \cos 2t \\ 9c_1 e^{-3t} - 4c_2 \cos 2t - 4c_3 \sin 2t \end{pmatrix}$, which is the same as that found in Exercise 9.1.2.

9.1.28. Solve the second order coupled system of ordinary differential equations $\ddot{u} = \dot{u} + u - v$, $\ddot{v} = \dot{v} - u + v$, by converting it into a first order system involving four variables.

Solution: Setting $\mathbf{u}(t) = \begin{pmatrix} u(t) \\ \dot{u}(t) \\ v(t) \\ \dot{v}(t) \end{pmatrix}$, the first order system is $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 1 \end{pmatrix}$. The coefficient matrix has eigenvalues $-1, 0, 1, 2$ and eigenvectors $\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \\ -2 \end{pmatrix}$. Thus $\mathbf{u}(t) = \begin{pmatrix} c_1 e^{-t} + c_2 + c_3 e^{t} + c_4 e^{2t} \\ -c_1 e^{-t} + c_3 e^{t} + 2c_4 e^{2t} \\ -c_1 e^{-t} + c_2 + c_3 e^{t} - c_4 e^{2t} \\ c_1 e^{-t} + c_3 e^{t} - 2c_4 e^{2t} \end{pmatrix}$, whose first and third components give the general solution $u(t) = c_1 e^{-t} + c_2 + c_3 e^{t} + c_4 e^{2t}$, $v(t) = -c_1 e^{-t} + c_2 + c_3 e^{t} - c_4 e^{2t}$ to the second order system.

9.1.29. Suppose that $\mathbf{u}(t) \in \mathbb{R}^n$ is a polynomial solution to the constant coefficient linear system $\dot{\mathbf{u}} = A\mathbf{u}$. What is the maximal possible degree of $\mathbf{u}(t)$? What can you say about A when $\mathbf{u}(t)$ has maximal degree?

Solution: The degree is at most $n-1$, and this occurs if and only if A has only one Jordan chain in its Jordan basis.

$\diamondsuit$ 9.1.30. (*a*) Under the assumption that $\mathbf{u}_1, \ldots, \mathbf{u}_k$ form a Jordan chain for the coefficient matrix A, prove that the functions (9.15) are solutions to the differential equation $\dot{\mathbf{u}} = A\mathbf{u}$. (*b*) Prove that they are linearly independent.

Solution:

(*a*) By direct computation,

$$\frac{d\mathbf{u}_j}{dt} = \lambda e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\,\mathbf{w}_i + e^{\lambda t} \sum_{i=1}^{j-1} \frac{t^{j-i-1}}{(j-i-1)!}\,\mathbf{w}_i,$$

which equals

$$A\mathbf{u}_j = e^{\lambda t} \sum_{i=1}^{j} \frac{t^{j-i}}{(j-i)!}\,A\mathbf{w}_i = e^{\lambda t}\left[\frac{t^{j-1}}{(j-1)!}\,\mathbf{w}_1 + \sum_{i=2}^{j} \frac{t^{j-i}}{(j-i)!}\,(\lambda\mathbf{w}_i + \mathbf{w}_{i-1})\right].$$

(*b*) At $t = 0$, we have $\mathbf{u}_j(0) = \mathbf{w}_j$, and the Jordan chain vectors are linearly independent.

9.1.31. (*a*) Explain how to solve the inhomogeneous ordinary differential equation $\dfrac{d\mathbf{u}}{dt} = A\mathbf{u} + \mathbf{b}$ when $\mathbf{b}$ is a constant vector belonging to $\operatorname{rng} A$. *Hint*: Look at $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}^\star$ where $\mathbf{u}^\star$ is an equilibrium solution. (*b*) Use your method to solve

(*i*) $\dfrac{du}{dt} = u - 3v + 1$, $\dfrac{dv}{dt} = -u - v$, (*ii*) $\dfrac{du}{dt} = 4v + 2$, $\dfrac{dv}{dt} = -u - 3$.

Solution:

(*a*) The equilibrium solution satisfies $A\mathbf{u}^\star = -\mathbf{b}$, and so $\mathbf{v}(t) = \mathbf{u}(t) - \mathbf{u}^\star$ satisfies $\dot{\mathbf{v}} = \dot{\mathbf{u}} = A\mathbf{u} + \mathbf{b} = A(\mathbf{u} - \mathbf{u}^\star) = A\mathbf{v}$, which is the homogeneous system.

(b) (i) $u(t) = -3 c_1 e^{2t} + c_2 e^{-2t} - \frac{1}{4}$, $v(t) = c_1 e^{2t} + c_2 e^{-2t} + \frac{1}{4}$. (ii) $u(t) = -2 c_1 \cos 2t + 2 c_2 \sin 2t - 3$, $v(t) = c_1 \sin 2t + c_2 \cos 2t - \frac{1}{2}$.

stability

9.2. Stability of Linear Systems.

9.2.1. Classify the following systems according to whether the origin is (i) asymptotically stable, (ii) stable, or (iii) unstable: (a) $\frac{du}{dt} = -2u - v$, $\frac{dv}{dt} = u - 2v$. (b) $\frac{du}{dt} = 2u - 5v$, $\frac{dv}{dt} = u - v$. (c) $\frac{du}{dt} = -u - 2v$, $\frac{dv}{dt} = 2u - 5v$. (d) $\frac{du}{dt} = -2v$, $\frac{dv}{dt} = 8u$. (e) $\frac{du}{dt} = -2u - v + w$, $\frac{dv}{dt} = -u - 2v + w$, $\frac{dw}{dt} = -3u - 3v + 2w$. (f) $\frac{du}{dt} = -u - 2v$, $\frac{dv}{dt} = 6u + 3v - 4w$, $\frac{dw}{dt} = 4u - 3w$. (g) $\frac{du}{dt} = 2u - v + 3w$, $\frac{dv}{dt} = u - v + w$, $\frac{dw}{dt} = -4u + v - 5w$. (h) $\frac{du}{dt} = u + v - w$, $\frac{dv}{dt} = -2u - 3v + 3w$, $\frac{dw}{dt} = -v + w$.

Solution:
(a) asymptotically stable — eigenvalues $-2 \pm \mathrm{i}$,
(b) unstable — eigenvalues $\frac{1}{2} \pm \frac{\sqrt{11}}{2}\,\mathrm{i}$,
(c) asymptotically stable — eigenvalue -3
(d) stable — eigenvalues $\pm 4\,\mathrm{i}$
(e) stable — eigenvalues $0, -1$, with 0 complete
(f) unstable — eigenvalues $1, -1 \pm 2\,\mathrm{i}$,
(g) asymptotically stable — eigenvalues $-1, -2$,
(h) unstable — eigenvalues $-1, 0$, with 0 incomplete.

9.2.2. Write out the formula for the general real solution to the system in Example 9.16 and verify its stability.

Solution:

$$u(t) = e^{-t}\Big[\Big(c_1 + \sqrt{\tfrac{2}{3}}\,c_2\Big)\cos\sqrt{6}\,t + \Big(-\sqrt{\tfrac{2}{3}}\,c_1 + c_2\Big)\sin\sqrt{6}\,t\Big] + \tfrac{1}{2}\,c_3 e^{-2t},$$
$$v(t) = e^{-t}\Big[c_1\cos\sqrt{6}\,t + c_2\sin\sqrt{6}\,t\Big] + \tfrac{1}{2}\,c_3 e^{-2t},$$
$$w(t) = e^{-t}\Big[c_1\cos\sqrt{6}\,t + c_2\sin\sqrt{6}\,t\Big] + c_3 e^{-2t}.$$

9.2.3. Write out and solve the gradient flow system corresponding to the following quadratic forms: (a) $u^2 + v^2$, (b) uv, (c) $4u^2 - 2uv + v^2$, (d) $2u^2 - uv - 2uw + 2v^2 - vw + 2w^2$.

Solution:
(a) $\dot u = -2u$, $\dot v = -2v$, with solution $u(t) = c_1 e^{-2t}$, $v(t) = c_2 e^{-2t}$.
(b) $\dot u = -v$, $\dot v = -u$, with solution $u(t) = c_1 e^{t} + c_2 e^{-t}$, $v(t) = -c_1 e^{t} + c_2 e^{-t}$.
(c) $\dot u = -8u + 2v$, $\dot v = 2u - 2v$, with solution $u(t) = -c_1 \frac{\sqrt{13}+3}{2}\, e^{-(5+\sqrt{13})t} + c_2 \frac{\sqrt{13}-3}{2}\, e^{-(5-\sqrt{13})t}$, $v(t) = c_1 e^{-(5+\sqrt{13})t} + c_2 e^{-(5-\sqrt{13})t}$.
(d) $\dot u = -4u + v + 2v$, $\dot v = u - 4v + w$, $\dot w = 2u + v - 4w$, with solution $u(t) = -c_1 e^{-6t} + c_2 e^{-(3+\sqrt{3})t} + c_3 e^{-(3-\sqrt{3})t}$, $v(t) = -(\sqrt{3}+1)\,c_2 e^{-(3+\sqrt{3})t} + (\sqrt{3}-1)\,c_3 e^{-(3-\sqrt{3})t}$, $w(t) = c_1 e^{-6t} + c_2 e^{-(3+\sqrt{3})t} + c_3 e^{-(3-\sqrt{3})t}$.

9.2.4. Write out and solve the Hamiltonian systems corresponding to the first three quadratic forms in Exercise 9.2.3. Which of them are stable?

Solution:

(a) $\dot u = 2v,\ \ \dot v = -2u$, with solution $u(t) = c_1 \cos 2t + c_2 \sin 2t, v(t) = -c_1 \sin 2t + c_2 \cos 2t$. Stable.

(b) $\dot u = u,\ \ \dot v = -v$, with solution $u(t) = c_1 e^t, v(t) = c_2 e^{-t}$. Unstable.

(c) $\dot u = -2u + 2v,\ \ \dot v = -8u + 2v$, with solution $u(t) = \frac14 (c_1 - \sqrt3\, c_2) \cos 2\sqrt3\, t + \frac14 (\sqrt3\, c_1 + c_2) \sin 2\sqrt3\, t, v(t) = c_1 \cos 2\sqrt3\, t + c_2 \sin 2\sqrt3\, t$. Stable.

9.2.5. Which of the following 2×2 systems are gradient flows? Which are Hamiltonian systems? In each case, discuss the stability of the zero solution.

(a) $\begin{aligned}\dot u &= -2u + v,\\ \dot v &= u - 2v,\end{aligned}$ (b) $\begin{aligned}\dot u &= u - 2v,\\ \dot v &= -2u + v,\end{aligned}$ (c) $\begin{aligned}\dot u &= v,\\ \dot v &= u,\end{aligned}$ (d) $\begin{aligned}\dot u &= -v,\\ \dot v &= u,\end{aligned}$ (e) $\begin{aligned}\dot u &= -u - 2v,\\ \dot v &= -2u - v.\end{aligned}$

Solution: (a) Gradient flow. Asymptotically stable. (b) Neither. Unstable. (c) Hamiltonian flow. Unstable. (d) Hamiltonian flow. Stable. (e) Neither. Unstable.

9.2.6. *True or false*: A nonzero linear 2×2 gradient flow cannot be a Hamiltonian flow.

Solution: True. If $K = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, then we must have $\dfrac{\partial H}{\partial v} = a\,u + b\,v$, $\dfrac{\partial H}{\partial u} = -b\,u - c\,v$. Therefore, by equality of mixed partials, $\dfrac{\partial^2 H}{\partial u\,\partial v} = a = -c$. But if $K > 0$, both diagonal entries must be positive, $a, c > 0$, which is a contradiction.

9.2.7. (a) Show that the matrix $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ has $\lambda = \pm\,\mathrm{i}$ as incomplete complex conjugate eigenvalues. (b) Find the general real solution to $\dot{\mathbf u} = A\mathbf u$. (c) Explain the behavior of a typical solution. Why is the zero solution not stable?

Solution:

(a) The characteristic equation is $\lambda^4 + 2\lambda^2 + 1 = 0$, and so $\pm\,\mathrm{i}$ are double eigenvalues. However, each has only one linearly independent eigenvector, namely $(1, \pm\,\mathrm{i}, 0, 0)^T$. The general solution is $\mathbf u(t) = \begin{pmatrix} c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \cos t \\ -c_1 \sin t + c_2 \cos t - c_3 t \sin t + c_4 t \cos t \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{pmatrix}$.

(b) All solutions with $c_3^2 + c_4^2 \neq 0$ spiral off to ∞ as $t \to \pm\infty$. Since these can start out arbitrarily close to $\mathbf 0$, the zero solution is not stable.

9.2.8. Let A be a real 3×3 matrix, and assume that the linear system $\dot{\mathbf u} = A\,\mathbf u$ has a periodic solution of period P. Prove that every periodic solution of the system has period P. What other types of solutions can there be? Is the zero solution necessarily stable?

Solution: Every solution to a real first order system of period P comes from complex conjugate eigenvalues $\pm 2\pi\,\mathrm{i}/P$. A 3×3 real matrix has at least one real eigenvalue λ_1. Therefore, if the system has a solution of period P, its eigenvalues are λ_1 and $\pm 2\pi\,\mathrm{i}/P$. If $\lambda_1 = 0$, every solution has period P. Otherwise, the solutions with no component in the direction of the real eigenvector all have period P, and are the only periodic solutions, proving the result. The system is stable (but never asymptotically stable) if and only if the real eigenvalue $\lambda_1 \leq 0$.

9.2.9. Are the conclusions of Exercise 9.2.8 valid when A is a 4×4 matrix?

Solution: No, since a 4×4 matrix could have two distinct complex conjugate pairs of purely imaginary eigenvalues, $\pm 2\pi\,\mathrm{i}\,/P_1, \pm 2\pi\,\mathrm{i}\,/P_2$, and would then have periodic solutions of periods P_1 and P_2. The general solution in such a case is quasi-periodic; see Section 9.5 for details.

9.2.10. Let A be a real 5×5 matrix, and assume that A has eigenvalues $\mathrm{i}, -\mathrm{i}, -2, -1$ (and no others). Is the zero solution to the linear system $\dot{\mathbf{x}} = A\,\mathbf{x}$ necessarily stable? Explain. Does your answer change if A is 6×6?

Solution: The system is stable since $\pm\,\mathrm{i}$ must be simple eigenvalues since a 5×5 matrix has 5 eigenvalues counting multiplicities, and the multiplicities of complex conjugate eigenvalues are the same. A 6×6 system can have a complex conjugate pair of incomplete double eigenvalues $\pm\,\mathrm{i}$ in addition to the simple real eigenvalues $-1, -2$, and in that case the origin is unstable.

9.2.11. *True or false*: The system $\dot{\mathbf{u}} = -H_n\,\mathbf{u}$, where H_n is the $n \times n$ Hilbert matrix (1.69), is asymptotically stable.

Solution: True since $H_n > 0$ by Proposition 3.34.

9.2.12. *True or false*: If K is positive semi-definite, then the zero solution to $\dot{\mathbf{u}} = -K\,\mathbf{u}$ is stable.

Solution: True, because all eigenvalues, including 0, of a symmetric matrix are complete.

9.2.13. Let $\mathbf{u}(t)$ solve $\dot{\mathbf{u}} = A\,\mathbf{u}$. Let $\mathbf{v}(t) = \mathbf{u}(-t)$ be its time reversal. (*a*) Write down the linear system $\dot{\mathbf{v}} = B\,\mathbf{v}$ satisfied by $\mathbf{v}(t)$. Then classify the following statements as *true* or *false*. Explain your answers. (*b*) If $\dot{\mathbf{u}} = A\,\mathbf{u}$ is asymptotically stable, then $\dot{\mathbf{v}} = B\,\mathbf{v}$ is unstable. (*c*) If $\dot{\mathbf{u}} = A\,\mathbf{u}$ is unstable, then $\dot{\mathbf{v}} = B\,\mathbf{v}$ is asymptotically stable. (*d*) If $\dot{\mathbf{u}} = A\,\mathbf{u}$ is stable, then $\dot{\mathbf{v}} = B\,\mathbf{v}$ is stable.

Solution: (*a*) $\dot{\mathbf{v}} = B\,\mathbf{v} = -A\,\mathbf{v}$. (*b*) True, since the eigenvalues of $B = -A$ are minus the eigenvalues of A, and so will all have positive real parts. (*c*) False. For example, a saddle point, with one positive and one negative eigenvalue is still unstable when going backwards in time. (*d*) False, unless all the eigenvalues of A and hence B are complete and purely imaginary or zero.

9.2.14. *True or false*: If A is a symmetric matrix, then the system $\dot{\mathbf{u}} = -A^2\,\mathbf{u}$ has an asymptotically stable equilibrium solution.

Solution: The eigenvalues of $-A^2$ are all of the form $-\lambda^2 < 0$ where λ is an eigenvalue of A. Thus, if A is nonsingular, the result is true, while if A is singular, then the equilibrium solutions are stable, but not asymptotically stable.

9.2.15. *True or false*: (*a*) If $\operatorname{tr} A > 0$, then the system $\dot{\mathbf{u}} = A\,\mathbf{u}$ is unstable. (*b*) If $\det A > 0$, then the system $\dot{\mathbf{u}} = A\,\mathbf{u}$ is unstable.

Solution: (*a*) True, since the sum of the eigenvalues equals the trace, so at least one must be positive or have positive real part in order that the trace be positive. (*b*) False. $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$ is an example of a asymptotically stable system with positive determinant.

9.2.16. Consider the differential equation $\dot{\mathbf{u}} = -K\,\mathbf{u}$, where K is positive semi-definite. (*a*) Find all equilibrium solutions. (*b*) Prove that all non-constant solutions decay exponentially fast to some equilibrium. What is the decay rate? (*c*) Is the origin (*i*) stable, (*ii*) asymptotically stable, or (*iii*) unstable? (*d*) Prove that, as $t \to \infty$, the solution $\mathbf{u}(t)$ converges to the orthogonal projection of its initial vector $\mathbf{a} = \mathbf{u}(0)$ onto $\ker K$.

Solution:

(*a*) Every $\mathbf{v} \in \ker K$ gives an equilibrium solution $\mathbf{u}(t) \equiv \mathbf{v}$.

(b) The general solution has the form

$$\mathbf{u}(t) = c_1 e^{-\lambda_1 t}\mathbf{v}_1 + \cdots + c_r e^{-\lambda_r t}\mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n,$$

where $\lambda_1, \ldots, \lambda_r > 0$ are the positive eigenvalues of K with (orthogonal) eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$, while $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ form a basis for the null eigenspace, i.e., $\ker K$. Thus, as $t \to \infty$, $\mathbf{u}(t) \to c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n \in \ker K$, which is an equilibrium solution.

(c) The origin is asymptotically stable if K is positive definite, and stable if K is positive semi-definite.

(d) Note that

$$\mathbf{a} = \mathbf{u}(0) = c_1 \mathbf{v}_1 + \cdots + c_r \mathbf{v}_r + c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n.$$

Since the eigenvectors are orthogonal, $c_{r+1}\mathbf{v}_{r+1} + \cdots + c_n \mathbf{v}_n$ is the orthogonal projection of $\mathbf{a}$ onto $\ker K$.

9.2.17. (a) Let $H(u,v) = a u^2 + b\; uv + c v^2$ be a quadratic function. Prove that the non-equilibrium trajectories of the associated Hamiltonian system and those of the gradient flow are mutually orthogonal, i.e., they always intersect at right angles. Verify this result for (i) $u^2 + 3v^2$, (ii) uv by drawing representative trajectories of both systems on the same graph.

Solution:

(a) The tangent to the Hamiltonian trajectory at a point $(u,v)^T$ is $\mathbf{v} = (\partial H/\partial v, -\partial H/\partial u)^T$ while the tangent to the gradient flow trajectory is $\mathbf{w} = (\partial H/\partial u, \partial H/\partial v)^T$. Since $\mathbf{v}\cdot\mathbf{w} = 0$, the tangents are orthogonal.

(b) ■

9.2.18. *True or false*: If the Hamiltonian system for $H(u,v)$ is stable, then the corresponding gradient flow $\dot{\mathbf{u}} = -\nabla H$ is stable.

Solution: False. Only the positive definite Hamiltonians lead to stable gradient flows.

9.2.19. Suppose that $\mathbf{u}(t)$ satisfies the gradient flow system (9.20). (a) Prove that $\dfrac{d}{dt}\, q(\mathbf{u}) = -\,\|\, K\mathbf{u}\,\|^2$. ($b$) Explain why if $\mathbf{u}(t)$ is any nonconstant solution to the gradient flow, then $q(\mathbf{u}(t))$ is a strictly decreasing function of t, thus quantifying how fast a gradient flow decreases energy.

Solution:

(a) When $q(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T K \mathbf{u}$ then

$$\frac{d}{dt}\, q(\mathbf{u}) = \tfrac{1}{2}\dot{\mathbf{u}}^T K\mathbf{u} + \tfrac{1}{2}\mathbf{u}^T K\dot{\mathbf{u}} = \dot{\mathbf{u}}^T K\mathbf{u} = -\,(K\mathbf{u})^T K\mathbf{u} = -\,\|\, K\mathbf{u}\,\|^2.$$

(b) Since $K\mathbf{u} \neq \mathbf{0}$ for any $\mathbf{u} \neq \mathbf{0}$, assuming $\mathbf{u}$ is not the equilibrium solution, $\dfrac{d}{dt}\, q(\mathbf{u}) = -\,\|\, K\mathbf{u}\,\|^2 < 0$ and hence $q(\mathbf{u})$ is a decreasing function of t. Indeed, for $K > 0$, the solution $\mathbf{u} \to \mathbf{0}$, and $q(\mathbf{u}) \to 0$ exponentially fast.

♡ 9.2.20. The law of *conservation of energy* states that the energy in a Hamiltonian system is constant on solutions. (a) Prove that if $\mathbf{u}(t)$ satisfies the Hamiltonian system (9.21), then $H(\mathbf{u}(t)) = c$ is a constant, and hence solutions $\mathbf{u}(t)$ move along the level sets of the Hamiltonian or energy function. Explain how the value of c is determined by the initial conditions. (b) Plot the level sets of H in the particular case $H(u,v) = u^2 - 2uv + 2v^2$ and verify that they coincide with the solution trajectories.

Solution:

(a) By the multivariable calculus chain rule

$$\frac{d}{dt}\,H(u(t),v(t)) = \frac{\partial H}{\partial u}\,\frac{du}{dt} + \frac{\partial H}{\partial v}\,\frac{dv}{dt} = \frac{\partial H}{\partial u}\,\frac{\partial H}{\partial v} + \frac{\partial H}{\partial v}\left(-\,\frac{\partial H}{\partial u}\right) \equiv 0.$$

Since $H(u(t),v(t)) \equiv c$, its value is $c = H(u_0,v_0)$ where $u(t_0) = u_0, v(t_0) = v_0$ are the initial conditions.

(b) ■

♢ 9.2.21. Prove Lemma 9.14.

Solution: Let us do the case $f(t) = t^k\,e^{\mu t}\cos \nu t$; replacing the cosine by a sine is a trivial modification of the proof. If $\mu > 0$ then $|\,f(t)\,| \to \infty$ since $e^{\mu t} \to \infty$, $t^k \to \infty$ for $k > 0$ and is equal to 1 for $k = 0$, while cosine term is bounded $|\cos \nu t\,| \le 1$. If $\mu = 0$, then $|\,f(t)\,| \le 1$ when $k = 0$, while $|\,f(t)\,| \to \infty$ if $k > 0$. If $\mu < 0$, then $|\,f(t)\,| \le t^k\,e^{\mu t} = e^{\mu t + k \log t} \to 0$ as $t \to \infty$, since $\mu t + k \log t \to -\infty$ when $\mu < 0$. *Q.E.D.*

♢ 9.2.22. Prove Proposition 9.17.

Solution: An eigensolution $\mathbf{u}(t) = e^{\lambda t}\mathbf{v}$ with $\lambda = \mu + \mathrm{i}\,\nu$ is bounded by $\|\,\mathbf{u}(t)\,\| \le \|\,\mathbf{v}\,\| e^{\mu t}$. Moreover, since exponentials grow faster than polynomials, any solution of the form $\mathbf{u}(t) = e^{\lambda t}\,\mathbf{p}(t)$ where $\mathbf{p}(t)$ is a vector of polynomials can be bounded by $C\,e^{a t}$ for any $a > \mu = \mathrm{Re}\,\lambda$ and some $C > 0$. Since every solution can be written as a linear combination of such solutions, every term is bounded by a multiple of $e^{a t}$ provided $a > a^\star = \max \mathrm{Re}\,\lambda$ and so, by the triangle inequality, is their sum. If the maximal eigenvalues are complete, then there are no polynomial terms, and we can use the eigensolution bound, so we can set $a = a^\star$. *Q.E.D.*

2dode 9.3. Two-Dimensional Systems.

9.3.1. For each the following: (*a*) Write the system as $\dot{\mathbf{u}} = A\mathbf{u}$. (*b*) Find the eigenvalues and eigenvectors of A. (*c*) Find the general real solution of the system. (*d*) Draw the phase portrait, indicating its type and stability properties: (*i*) $\dot u_1 = -u_2$, $\dot u_2 = 9\,u_1$, (*ii*) $\dot u_1 = 2\,u_1 - 3\,u_2$, $\dot u_2 = u_1 - u_2$, (*iii*) $\dot u_1 = 3\,u_1 - 2\,u_2$, $\dot u_2 = 2\,u_1 - 2\,u_2$.

Solution:

(*i*) $A = \begin{pmatrix} 0 & -1 \\ 9 & 0 \end{pmatrix}$; $\lambda_1 = 3\,\mathrm{i}$, $\mathbf{v}_1 = \begin{pmatrix} \mathrm{i} \\ 3 \end{pmatrix}$, $\lambda_2 = -3\,\mathrm{i}$, $\mathbf{v}_2 = \begin{pmatrix} -\,\mathrm{i} \\ 3 \end{pmatrix}$,

$u_1(t) = c_1\,\cos 3\,t + c_2 \sin 3\,t$, $u_2(t) = 3\,c_1\,\sin 3\,t - 3\,c_2\cos 3\,t$; center.

(*ii*) $A = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}$; $\lambda_1 = \frac12 + \mathrm{i}\,\frac{\sqrt3}{2}$, $\mathbf{v}_1 = \begin{pmatrix} \frac32 + \mathrm{i}\,\frac{\sqrt3}{2} \\ 1 \end{pmatrix}$, $\lambda_2 = \frac12 - \mathrm{i}\,\frac{\sqrt3}{2}$, $\mathbf{v}_2 = \begin{pmatrix} \frac32 - \mathrm{i}\,\frac{\sqrt3}{2} \\ 1 \end{pmatrix}$,

$u_1(t) = e^{-t/2}\left[\left(\frac32 c_1 - \frac{\sqrt3}{2} c_2\right)\cos \frac{\sqrt3}{2}\,t + \left(\frac{\sqrt3}{2} c_1 + \frac32 c_2\right)\sin \frac{\sqrt3}{2}\,t\right]$,

$u_2(t) = e^{-t/2}\left[c_1 \cos \frac{\sqrt3}{2}\,t + c_2 \sin \frac{\sqrt3}{2}\,t\right]$; stable focus

(*iii*) $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$, $\lambda_1 = -1$, $\lambda_2 = 2.$, $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$,

$u_1(t) = c_1\,e^{-t} + 2\,c_2\,e^{2t}$, $u_2(t) = 2\,c_1\,e^{-t} + c_2\,e^{2t}$; saddle point.

9.3.2. For each of the following systems

$$(i)\ \ \dot{\mathbf{u}} = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}\mathbf{u}, \qquad (ii)\ \ \dot{\mathbf{u}} = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}\mathbf{u}, \qquad (iii)\ \ \dot{\mathbf{u}} = \begin{pmatrix} -3 & 5/2 \\ -5/2 & 2 \end{pmatrix}\mathbf{u}.$$

(*a*) Find the general real solution. (*b*) Using the solution formulas obtained in part (*a*), plot several trajectories of each system. On your graphs, identify the eigenlines (if relevant), the direction of increasing t on the trajectories. (*c*) Write down the type and stability properties of the system.

Solution:

(*i*) $\mathbf{u}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; saddle point.

(*ii*) $\mathbf{u}(t) = c_1 e^{-t}\begin{pmatrix} 2\cos t - \sin t \\ 5\cos t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 2\sin t + \cos t \\ 5\sin t \end{pmatrix}$; stable focus.

(*iii*) $\mathbf{u}(t) = c_1 e^{-t/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t/2}\begin{pmatrix} t \\ t + \frac{2}{5} \end{pmatrix}$; stable improper node.

9.3.3. Classify the following systems, and sketch their phase portraits.

(a) $\dfrac{du}{dt} = -u + 4v, \quad \dfrac{dv}{dt} = u - 2v.$ (b) $\dfrac{du}{dt} = -2u + v, \quad \dfrac{dv}{dt} = u - 4v.$ (c) $\dfrac{du}{dt} = 5u + 4v, \quad \dfrac{dv}{dt} = u + 2v.$ (d) $\dfrac{du}{dt} = -3u - 2v, \quad \dfrac{dv}{dt} = 3u + 2v.$

Solution:

(a) For the matrix $A = \begin{pmatrix} -1 & 4 \\ 1 & -2 \end{pmatrix}$, $\operatorname{tr} A = -3 < 0$, $\det A = -2 < 0$, $\Delta = 17 > 0$, so this is an unstable saddle point.

(*b*) For the matrix $A = \begin{pmatrix} -2 & 1 \\ 1 & -4 \end{pmatrix}$, $\operatorname{tr} A = -6 < 0$, $\det A = 7 > 0$, $\Delta = 8 > 0$, so this is a stable node.

(*c*) For the matrix $A = \begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$, $\operatorname{tr} A = 7 > 0$, $\det A = 6 > 0$, $\Delta = 25 > 0$, so this is an unstable node.

(*d*) For the matrix $A = \begin{pmatrix} -3 & -2 \\ 3 & 2 \end{pmatrix}$, $\operatorname{tr} A = -1 < 0$, $\det A = 0$, $\Delta = 1 > 0$, so this is a stable line.

9.3.4. Sketch the phase portrait for the following systems: (a) $\dot u_1 = u_1 - 3u_2, \quad \dot u_2 = -3u_1 + u_2.$

(b) $\dot u_1 = 3u_1 - 4u_2, \quad \dot u_2 = u_1 - u_2.$ (c) $\dot u_1 = u_1 + u_2, \quad \dot u_2 = 4u_1 - 2u_2.$ (d) $\dot u_1 = u_1 + u_2, \quad \dot u_2 = u_2.$ (e) $\dot u_1 = \frac{3}{2}u_1 + \frac{5}{2}u_2, \quad \dot u_2 = \frac{5}{2}u_1 - \frac{3}{2}u_2.$

Solution: ■

9.3.5. (a) Solve the initial value problem $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -1 & 2 \\ -1 & -3 \end{pmatrix}\mathbf{u}, \quad \mathbf{u}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$

(*b*) Sketch a picture of your solution curve $\mathbf{u}(t)$, indicating the direction of motion.

(*c*) Is the origin (*i*) stable? (*ii*) asymptotically stable? (*iii*) unstable? (*iv*) none of these? Justify your answer.

Solution: (a) $\mathbf{u}(t) = \left(e^{-2t}\cos t + 7e^{-2t}\sin t, 3e^{-2t}\cos t - 4e^{-2t}\sin t \right)^T$; (b) ■ (*c*) asymptotically stable since the coefficient matrix has $\operatorname{tr} A = -4 < 0$, $\det A = 5 > 0$, $\Delta = -4 < 0$, and hence it is a stable focus, or, equivalently, the eigenvalues $-2 \pm \mathrm{i}$ have negative real part.

♢ 9.3.6. Justify the solution formulas (9.30) and (9.31).

Solution: For (9.30), the complex solution

$$e^{\lambda t}\mathbf{v} = e^{(\mu + \mathrm{i}\,\nu)t}(\mathbf{w} + \mathrm{i}\,\mathbf{z}) = e^{\mu t}\big[\cos(\nu t)\,\mathbf{w} - \sin(\nu t)\,\mathbf{z}\big] + e^{\mu t}\big[\sin(\nu t)\,\mathbf{w} + \cos(\nu t)\,\mathbf{z}\big]$$

leads to the general real solution

$$\begin{aligned}\mathbf{u}(t) &= c_1 e^{\mu t}\big[\cos(\nu t)\,\mathbf{w} - \sin(\nu t)\,\mathbf{z}\big] + c_2 e^{\mu t}\big[\sin(\nu t)\,\mathbf{w} + \cos(\nu t)\,\mathbf{z}\big] \\ &= e^{\mu t}\big[c_1\cos(\nu t) + c_2\sin(\nu t)\big]\,\mathbf{w} + e^{\mu t}\big[-c_1\sin(\nu t) + c_2\cos(\nu t)\big]\,\mathbf{z} \\ &= r\,e^{\mu t}\big[\cos(\nu t - \sigma)\,\mathbf{w} + \sin(\nu t - \sigma)\,\mathbf{z}\big],\end{aligned}$$

where $r = \sqrt{c_1^2 + c_2^2}$ and $\tan\sigma = c_2/c_1$.

To justify (9.31), we differentiate

$$\frac{d\mathbf{u}}{dt} = \frac{d}{dt}\big[(c_1 + c_2 t)e^{\lambda t}\,\mathbf{v} + c_2 e^{\lambda t}\,\mathbf{w}\big] = \lambda\big[(c_1 + c_2 t)e^{\lambda t}\,\mathbf{v} + c_2 e^{\lambda t}\,\mathbf{w}\big] + c_2 e^{\lambda t}\,\mathbf{v},$$

while

$$A\mathbf{u} = (c_1 + c_2 t)e^{\lambda t} A\mathbf{v} + c_2 e^{\lambda t} A\mathbf{w} = (c_1 + c_2 t)e^{\lambda t}\,\lambda\mathbf{v} + c_2 e^{\lambda t}(\lambda\mathbf{w} + \mathbf{v})$$

by the Jordan chain condition. Therefore $\dot{\mathbf{u}} = A\mathbf{u}$.

9.3.7. Which of the 14 possible two-dimensional phase portraits can occur for the phase plane equivalent (9.7) of a second order scalar ordinary differential equation?

Solution: All except for IV(a–c), the stars and the trivial case.

ME 9.4. Matrix Exponentials.

9.4.1. Find the exponentials e^{tA} of the following 2×2 matrices:

(a) $\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, (e) $\begin{pmatrix} -1 & 2 \\ -5 & 5 \end{pmatrix}$, (f) $\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}$.

Solution:

(a) $\begin{pmatrix} \frac{4}{3}e^t - \frac{1}{3}e^{-2t} & -\frac{1}{3}e^t + \frac{1}{3}e^{-2t} \\ \frac{4}{3}e^t - \frac{4}{3}e^{-2t} & -\frac{1}{3}e^t + \frac{4}{3}e^{-2t} \end{pmatrix}$; (b) $\begin{pmatrix} \frac{1}{2}e^t + \frac{1}{2}e^{-t} & \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^t - \frac{1}{2}e^{-t} & \frac{1}{2}e^t + \frac{1}{2}e^{-t} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$;

(c) $\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$; (d) $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$; (e) $\begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix}$;

(f) $\begin{pmatrix} e^{-t} + 2te^{-t} & 2te^{-t} \\ -2te^{-t} & e^{-t} - 2te^{-t} \end{pmatrix}$.

9.4.2. Determine the matrix exponential e^{tA} for the following matrices:

(a) $\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$, (c) $\begin{pmatrix} -1 & 1 & 1 \\ -2 & -2 & -2 \\ 1 & -1 & -1 \end{pmatrix}$, (d) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

Solution:

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix}$;

(b) $\begin{pmatrix} \frac{1}{6}e^t + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^t - \frac{1}{3}e^{4t} & \frac{1}{6}e^t - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \\ \frac{1}{3}e^t - \frac{1}{3}e^{4t} & \frac{2}{3}e^t + \frac{1}{3}e^{4t} & \frac{1}{3}e^t - \frac{1}{3}e^{4t} \\ \frac{1}{6}e^t - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^t - \frac{1}{3}e^{4t} & \frac{1}{6}e^t + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \end{pmatrix}$;

(c) $\begin{pmatrix} e^{-2t}+te^{-2t} & te^{-2t} & te^{-2t} \\ -1+e^{-2t} & e^{-2t} & -1+e^{-2t} \\ 1-e^{-2t}-te^{-2t} & -te^{-2t} & 1-te^{-2t} \end{pmatrix}$;

(d) $$\begin{pmatrix} \frac13 e^t + \frac23 e^{-t/2}\cos\frac{\sqrt3}{2}t & \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t - \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t & \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t + \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t \\ \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t + \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t & \frac13 e^t + \frac23 e^{-t/2}\cos\frac{\sqrt3}{2}t & \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t - \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t \\ \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t - \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t & \frac13 e^t - \frac13 e^{-t/2}\cos\frac{\sqrt3}{2}t + \frac{1}{\sqrt3}e^{-t/2}\sin\frac{\sqrt3}{2}t & \frac13 e^t + \frac23 e^{-t/2}\cos\frac{\sqrt3}{2}t \end{pmatrix}.$$

9.4.3. Verify the determinant formula of Lemma 9.28 for the matrices in Exercises 9.4.1 and 9.4.2.

Solution: 9.4.1 (a) $\det e^{tA} = e^{-t} = e^{t\,\mathrm{tr}\,A}$, (b) $\det e^{tA} = 1 = e^{t\,\mathrm{tr}\,A}$, (c) $\det e^{tA} = 1 = e^{t\,\mathrm{tr}\,A}$, (d) $\det e^{tA} = 1 = e^{t\,\mathrm{tr}\,A}$, (e) $\det e^{tA} = e^{4t} = e^{t\,\mathrm{tr}\,A}$, (f) $\det e^{tA} = e^{-2t} = e^{t\,\mathrm{tr}\,A}$. 9.4.2 (a) $\det e^{tA} = 1 = e^{t\,\mathrm{tr}\,A}$, (b) $\det e^{tA} = e^{8t} = e^{t\,\mathrm{tr}\,A}$, (c) $\det e^{tA} = e^{-4t} = e^{t\,\mathrm{tr}\,A}$, (d) $\det e^{tA} = 1 = e^{t\,\mathrm{tr}\,A}$.

9.4.4. Find e^A when $A =$
(a) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{pmatrix}$, (e) $\begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$.

Solution: (a) $\begin{pmatrix} \frac12(e^3+e^7) & \frac12(e^3-e^7) \\ \frac12(e^3-e^7) & \frac12(e^3+e^7) \end{pmatrix}$, (b) $\begin{pmatrix} e\cos\sqrt2 & -\sqrt2\, e\sin\sqrt2 \\ \frac{1}{\sqrt2}\, e\sin\sqrt2 & e\cos\sqrt2 \end{pmatrix}$, (c) $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$,
(d) $\begin{pmatrix} e & 0 & 0 \\ 0 & e^{-2} & 0 \\ 0 & 0 & e^{-5} \end{pmatrix}$, (e) $\begin{pmatrix} \frac49 + \frac59\cos 3 & \frac49 - \frac49\cos 3 + \frac13\sin 3 & \frac29 - \frac29\cos 3 - \frac23\sin 3 \\ \frac49 - \frac49\cos 3 - \frac13\sin 3 & \frac49 + \frac59\cos 3 & \frac29 - \frac29\cos 3 + \frac23\sin 3 \\ \frac29 - \frac29\cos 3 + \frac23\sin 3 & \frac29 - \frac29\cos 3 - \frac23\sin 3 & \frac19 + \frac89\cos 3 \end{pmatrix}$.

9.4.5. Solve the indicated initial value problems by first exponentiating the coefficient matrix and then applying formula (9.40): (a) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,
(b) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} 3 & -6 \\ 4 & -7 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, (c) $\dfrac{d\mathbf{u}}{dt} = \begin{pmatrix} -9 & -6 & 6 \\ 8 & 5 & -6 \\ -2 & 1 & 3 \end{pmatrix}\mathbf{u}$, $\mathbf{u}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Solution:
(a) $\mathbf{u}(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \cos t + 2\sin t \\ \sin t - 2\cos t \end{pmatrix}$;
(b) $\mathbf{u}(t) = \begin{pmatrix} 3e^{-t} - 2e^{-3t} & -3e^{-t} + 3e^{-3t} \\ 2e^{-t} - 2e^{-3t} & -2e^{-t} + 3e^{-3t} \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -6e^{-t} + 5e^{-3t} \\ -4e^{-t} + 5e^{-3t} \end{pmatrix}$;

$$
(c)\quad \mathbf{u}(t) = \begin{pmatrix} 3e^{-t} - 2\cos 3t - 2\sin 3t & 3e^{-t} - 3\cos 3t - \sin 3t & 2\sin 3t \\ -2e^{-t} + 2\cos 3t + 2\sin 3t & -2e^{-t} + 3\cos 3t + \sin 3t & -2\sin 3t \\ 2e^{-t} - 2\cos 3t & 2e^{-t} - 2\cos 3t + \sin 3t & \cos 3t + \sin 3t \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3e^{-t} - 3\cos 3t - \sin 3t \\ -2e^{-t} + 3\cos 3t + \sin 3t \\ 2e^{-t} - 2\cos 3t + \sin 3t \end{pmatrix}.
$$

9.4.6. What is $e^{t\,\mathrm{O}}$ when O is the $n \times n$ zero matrix?

Solution: $e^{t\,\mathrm{O}} = \mathrm{I}$ for all t.

9.4.7. Find all matrices A such that $e^{tA} = \mathrm{O}$.

Solution: There are none, since e^{tA} is always invertible.

9.4.8. Let $A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$. Show that $e^A = \mathrm{I}$.

Solution:
$e^{tA} = \begin{pmatrix} \cos 2\pi t & -\sin 2\pi t \\ \sin 2\pi t & \cos 2\pi t \end{pmatrix}$, and hence, when $t = 1$, $e^A = \begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

9.4.9. (*a*) Let A be a 2×2 matrix such that $\operatorname{tr} A = 0$ and $\delta = \sqrt{\det A} > 0$. Prove that $e^A = (\cos \delta)\,\mathrm{I} + \dfrac{\sin \delta}{\delta}\,A$. *Hint*: Use Exercise 8.2.50. (*b*) Establish a similar formula when $\det A < 0$. (*c*) What if $\det A = 0$?

Solution: (a) According to Exercise 8.2.50, $A^2 = -\delta^2\,\mathrm{I}$ since $\operatorname{tr} A = 0$, $\det A = \delta^2$. Thus, by induction, $A^{2m} = (-1)^m \delta^{2m}\,\mathrm{I}$, $A^{2m+1} = (-1)^m \delta^{2m} A$.

$$
e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = \sum_{m=0}^{\infty} (-1)^m \frac{(\delta t)^{2m}}{(2m)!}\,\mathrm{I} + \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m+1}\,\delta^{2m}}{(2m+1)!}\,A = \cos \delta t + \frac{\sin \delta t}{\delta}.
$$

Setting $t = 1$ proves the formula. (*b*) $e^A = (\cosh \delta)\,\mathrm{I} + \dfrac{\sinh \delta}{\delta}\,A$ where $\det A = -\delta^2$.
(*c*) $e^A = \mathrm{I} + A$ since $A^2 = \mathrm{O}$ by Exercise 8.2.50.

♢ 9.4.10. Explain in detail why the columns of e^{tA} form a basis for the solution space to the system $\dot{\mathbf{u}} = A\,\mathbf{u}$.

Solution: Assuming A is an $n \times n$ matrix, since e^{tA} is a matrix solution, its n individual columns must be solutions. Moreover, the columns are linearly independent since $e^{0\,A} = \mathrm{I}$ is nonsingular. Therefore, they form a basis for the n-dimensional solution space.

9.4.11. *True or false*: (a) $e^{A^{-1}} = \left(e^A\right)^{-1}$, (*b*) $e^{A + A^{-1}} = e^A\, e^{A^{-1}}$.

Solution: (a) False, unless $A^{-1} = -A$. (*b*) True, since A and A^{-1} commute.

♢ 9.4.12. Prove formula (9.42). *Hint*: Fix s and prove that, as functions of t, both sides of the equation define matrix solutions with the same initial conditions. Then use uniqueness.

Solution: Fix s and let $U(t) = e^{(t+s)\,A}$, $V(t) = e^{tA}\,e^{sA}$. Then, by the chain rule, $\dot{U} = A\,e^{(t+s)\,A} = A\,U$, while, by the matrix Leibniz rule (9.39), $\dot{V} = A\,e^{tA}\,e^{sA} = A\,V$. Moreover, $U(0) = e^{sA} = V(0)$. Thus $U(t)$ and $V(t)$ solve the same initial value problem, hence, by uniqueness, $U(t) = V(t)$ for all t. *Q.E.D.*

9.4.13. Prove that A commutes with its exponential: $A\,e^{tA} = e^{tA}A$. *Hint*: Prove that both are matrix solutions to $\dot U = A\,U$ with the same initial conditions.

Solution: Set $U(t) = A\,e^{tA}, V(t) = e^{tA}\,A$. Then, by the matrix Leibniz formula (9.39), $\dot U = A^2\,e^{tA} = A\,U$, $\dot V = A\,e^{tA}\,A = A\,V$, while $U(0) = A = V(0)$. Thus $U(t)$ and $V(t)$ solve the same initial value problem, hence, by uniqueness, $U(t) = V(t)$ for all t. *Q.E.D.*

9.4.14. Prove that $e^{t(A-\lambda\,\mathrm{I})} = e^{-t\lambda}\,e^{tA}$ by showing that both sides are matrix solutions to the same initial value problem.

Solution: Set $U(t) = e^{-t\lambda}\,e^{tA}$. Then, $\dot U = -\lambda\,e^{-t\lambda}\,e^{tA} + e^{-t\lambda}\,A\,e^{tA} = (A - \lambda\,\mathrm{I})U$. Moreover, $U(0) = \mathrm{I}$. Therefore, by the the definition of matrix exponential, $U(t) = e^{t(A-\lambda\,\mathrm{I})}$.

♢ 9.4.15. (*a*) Prove that the exponential of the transpose of a matrix is the transpose of its exponential: $e^{tA^T} = (e^{tA})^T$. (*b*) What does this imply about the solutions to the linear systems $\dot{\mathbf{u}} = A\,\mathbf{u}$ and $\mathbf{v} = A^T\mathbf{v}$?

Solution: (a) Let $V(t) = (e^{tA})^T$. Then $\dfrac{dU}{dt} = \left(\dfrac{d}{dt}\,e^{tA}\right)^T = (e^{tA}\,A)^T = A^T(e^{tA})^T = A^T V$, and $V(0) = \mathrm{I}$. Therefore, by the the definition of matrix exponential, $V(t) = e^{tA^T}$.
(*b*) The columns of e^{tA} form a basis for the solutions to $\dot{\mathbf{u}} = A\,\mathbf{u}$, while its rows are a basis for the solutions to $\mathbf{v} = A^T\mathbf{v}$.

♢ 9.4.16. Prove that if $A = S\,B\,S^{-1}$ are similar matrices, then so are their exponentials: $e^{tA} = S\,e^{tB}\,S^{-1}$.

Solution: First note that $A^n = S\,B^n S^{-1}$. Therefore,

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,A^n = \sum_{n=0}^{\infty} \frac{t^n}{n!}\,S\,B^n S^{-1} = S\left(\sum_{n=0}^{\infty} \frac{t^n}{n!}\,B^n\right)S^{-1} = S\,e^{tB}\,S^{-1}.$$

An alternative proof uses the fact that e^{tA} and $S\,e^{tB}\,S^{-1}$ both satisfy the initial value problem $\dot U = AU = S\,B\,S^{-1}U$, $U(0) = \mathrm{I}$, and hence, by uniqueness, they are equal.

♢ 9.4.17. Diagonalization provides an alternative method for computing the exponential of a complete matrix. (a) First show that if $D = \operatorname{diag}(d_1, \ldots, d_n)$ is a diagonal matrix, so is $e^{tD} = \operatorname{diag}(e^{t\,d_1}, \ldots, e^{t\,d_n})$. (*b*) Second, prove that if $A = S\,D\,S^{-1}$ is diagonalizable, so is $e^{tA} = S\,e^{tD}\,S^{-1}$. (*c*) When possible, use diagonalization to compute the exponentials of the matrices in Exercises 9.4.1–2.

Solution: (a) $\dfrac{d}{dt}\operatorname{diag}(e^{t\,d_1}, \ldots e^{t\,d_n}) = \operatorname{diag}(d_1\,e^{t\,d_1}, \ldots, d_n\,e^{t\,d_n}) = D\operatorname{diag}(e^{t\,d_1}, \ldots, e^{t\,d_n})$. Moreover, at $t = 0$, we have $\operatorname{diag}(e^{0\,d_1}, \ldots, e^{0\,d_n}) = \mathrm{I}$. Therefore, $\operatorname{diag}(e^{t\,d_1}, \ldots, e^{t\,d_n})$ satisfies the defining properties of e^{tD}. (*b*) See Exercise 9.4.16. (*c*) 9.4.1:

(a) $$\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac43 e^t - \frac13 e^{-2t} & -\frac13 e^t + \frac13 e^{-2t} \\ \frac43 e^t - \frac43 e^{-2t} & -\frac13 e^t + \frac43 e^{-2t} \end{pmatrix};$$

(*b*) $$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac12 e^t + \frac12 e^{-t} & \frac12 e^t - \frac12 e^{-t} \\ \frac12 e^t - \frac12 e^{-t} & \frac12 e^t + \frac12 e^{-t} \end{pmatrix};$$

(*c*) $$\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}\begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix};$$

(*d*) not diagonalizable;

(*e*) $$\begin{pmatrix} \frac{3}{5} - \frac{1}{5}\,\mathrm{i} & \frac{3}{5} + \frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(2+\mathrm{i})t} & 0 \\ 0 & e^{(2-\mathrm{i})t} \end{pmatrix} \begin{pmatrix} \frac{3}{5} - \frac{1}{5}\,\mathrm{i} & \frac{3}{5} + \frac{1}{5}\,\mathrm{i} \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} e^{2t}\cos t - 3e^{2t}\sin t & 2e^{2t}\sin t \\ -5e^{2t}\sin t & e^{2t}\cos t + 3e^{2t}\sin t \end{pmatrix};$$

(*f*) not diagonalizable.

9.4.2:

(a) $$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\mathrm{i}t} & 0 \\ 0 & 0 & e^{-\mathrm{i}t} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\mathrm{i} & \mathrm{i} \\ 2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2\sin t & \cos t & \sin t \\ 2\cos t - 2 & -\sin t & \cos t \end{pmatrix};$$

(*b*) $$\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{4t} \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{1}{6}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \\ \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{2}{3}e^{t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} \\ \frac{1}{6}e^{t} - \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} & \frac{1}{3}e^{t} - \frac{1}{3}e^{4t} & \frac{1}{6}e^{t} + \frac{1}{2}e^{3t} + \frac{1}{3}e^{4t} \end{pmatrix};$$

(*c*) not diagonalizable;

(*d*) $$\begin{pmatrix} 1 & -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} & -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} & -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-(\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2})t} & 0 \\ 0 & 0 & e^{-(\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2})t} \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} & -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2} & -\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2} \\ 1 & 1 & 1 \end{pmatrix}^{-1},$$
which is the same as before.

♢ 9.4.18. Justify the matrix Leibniz rule (9.39) using the formula for matrix multiplication.

Solution: Let M have size $p \times q$ and N have size $q \times r$. The derivative of the (i,j) entry of $M(t)\,N(t)$ is

$$\frac{d}{dt} \sum_{k=1}^{q} m_{ik}(t)\, n_{kj}(t) = \sum_{k=1}^{q} \frac{dm_{ik}}{dt}\, n_{kj}(t) + \sum_{k=1}^{q} m_{ik}(t)\, \frac{dn_{kj}}{dt}.$$

The first sum is the (i,j) entry of $\dfrac{dM}{dt}\,N$ while the second is the (i,j) entry of $M\,\dfrac{dN}{dt}$. *Q.E.D.*

♢ 9.4.19. Let A be a real matrix. (*a*) Explain why e^{A} is a real matrix. (*b*) Prove that $\det e^{A} > 0$.

Solution: (*a*) The exponential series is a sum of real terms. Alternatively, one can choose a real basis for the solution space to construct the real matrix solution $U(t)$ before substituting into formula (9.41). (*b*) According to Lemma 9.28, $\det e^{A} = e^{\operatorname{tr} A} > 0$ since a real scalar exponential is always positive.

9.4.20. Show that $\operatorname{tr} A = 0$ if and only if $\det e^{tA} = 1$ for all t.

Solution: Lemma 9.28 implies $\det e^{tA} = e^{t \operatorname{tr} A} = 1$ if and only if $\operatorname{tr} A = 0$.

♢ 9.4.21. Prove that if λ is an eigenvalue of A, then $e^{t\lambda}$ is an eigenvalue of e^{tA}. What is the eigenvector?

Solution: Let $\mathbf{u}(t) = e^{t\lambda}\mathbf{v}$ where $\mathbf{v}$ is the eigenvector of A. Then, $\dfrac{d\mathbf{u}}{dt} = \lambda e^{t\lambda}\mathbf{v} = \lambda\mathbf{u} = A\mathbf{u}$, and hence, by (9.40), $\mathbf{u}(t) = e^{tA}\mathbf{u}(0) = e^{tA}\mathbf{v}$. Therefore, equating the two formulas for $\mathbf{u}(t)$, we conclude that $e^{tA}\mathbf{v} = e^{t\lambda}\mathbf{v}$, which proves that $\mathbf{v}$ is an eigenvector of e^{tA} with eigenvalue $e^{t\lambda}$.

9.4.22. Show that the origin is an asymptotically stable equilibrium solution to $\dot{\mathbf{u}} = A\mathbf{u}$ if and only if $\lim_{t\to\infty} e^{tA} = \mathrm{O}$.

Solution: The origin is an asymptotically stable if and only if all solutions tend to zero as $t \to \infty$. Thus, all columns of e^{tA} tend to $\mathbf{0}$ as $t \to \infty$, and hence $\lim_{t\to\infty} e^{tA} = \mathrm{O}$. Conversely, if $\lim_{t\to\infty} e^{tA} = \mathrm{O}$, then any solution has the form $\mathbf{u}(t) = e^{tA}\mathbf{c}$, and hence $\mathbf{u}(t) \to \mathbf{0}$ as $t \to \infty$, proving asymptotic stability.

9.4.23. Let A be a real square matrix and e^A its exponential. Under what conditions does the linear system $\dot{\mathbf{u}} = e^A\mathbf{u}$ have an asymptotically stable equilibrium solution?

Solution: According to Exercise 9.4.21, the eigenvalues of e^A are $e^\lambda = e^\mu \cos\nu + \mathrm{i}\, e^\mu \sin\nu$, where $\lambda = \mu + \mathrm{i}\,\nu$ are the eigenvalues of A. For e^λ to have negative real part, we must have $\cos\nu < 0$, and so $\nu = \operatorname{Im}\lambda$ must lie between $\left(2k + \frac{1}{2}\right)\pi < \nu < \left(2k + \frac{3}{2}\right)\pi$ for some $k \in \mathbb{Z}$.

9.4.24. Prove that if $U(t)$ is any matrix solution to $\dfrac{dU}{dt} = AU$, so is $\widetilde{U}(t) = U(t)\,C$ where C is any constant matrix (of compatible size).

Solution: Indeed, the columns of $\widetilde{U}(t)$ are linear combinations of the columns of $U(t)$, and hence automatically solutions to the linear system. Alternatively, we can prove this directly using the Leibniz rule (9.39). Therefore, $\dfrac{d\widetilde{U}}{dt} = \dfrac{d}{dt}\left[\,U(t)\,C\,\right] = \dfrac{dU}{dt}\,C = AUC = A\widetilde{U}$, since C is constant.

♢ 9.4.25. (*a*) Show that $U(t)$ satsifies the matrix differential equation $\dot{U} = UB$ if and only if $U(t) = C e^{tB}$ where $C = U(0)$.
(*b*) Show that if $U(0)$ is nonsingular, then $U(t)$ also satisfies a matrix differential equation of the form $\dot{U} = AU$. Is $A = B$? *Hint*: Use Exercise 9.4.16.

Solution:

(*a*) If $U(t) = C e^{tB}$, then $\dfrac{dU}{dt} = C e^{tB} B = UB$, and so U satisfies the differential equation. Moreover, $C = U(0)$. Thus, $U(t)$ is the unique solution to the initial value problem $\dot{U} = UB,\ U(0) = C$. *Q.E.D.*

(*b*) By Exercise 9.4.16, $U(t) = C e^{tB} = e^{tA} C$ where $A = CBC^{-1}$. Thus, $\dot{U} = AU$ as claimed. Note that $A = B$ if and only if A commutes with $U(0)$.

9.4.26. (*a*) Suppose $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$ are vector-valued functions whose values at each point t are linearly independent vectors in $\mathbb{R}^n$. Show that they form a basis for the solution space of a homogeneous constant coefficient linear system $\dot{\mathbf{u}} = A\mathbf{u}$ if and only if each $d\mathbf{u}_j/dt$ is a linear combination of $\mathbf{u}_1(t), \ldots, \mathbf{u}_n(t)$. (*b*) Show that if $\mathbf{u}(t)$ solves $\dot{\mathbf{u}} = A\mathbf{u}$, so do its derivatives $d^j\mathbf{u}/dt^j$, $j = 1, 2, \ldots$. (*c*) Show that a function $\mathbf{u}(t)$ belongs to the solution space of a homogeneous constant coefficient linear system $\dot{\mathbf{u}} = A\mathbf{u}$ if and only if $\dfrac{d^n\mathbf{u}}{dt^n}$ is a linear combination of $\mathbf{u}, \dfrac{d\mathbf{u}}{dt}, \ldots, \dfrac{d^{n-1}\mathbf{u}}{dt^{n-1}}$.

Solution:

(*a*) Let $U(t) = (\,\mathbf{u}_1(t)\ \ldots\ \mathbf{u}_n(t)\,)$ be the corresponding matrix-valued function. Then $\dfrac{d\mathbf{u}_j}{dt} =$

$\sum_{i=1}^{n} b_{ij} \mathbf{u}_i$ for all $j = 1, \ldots, n$, if and only if $\dot{U} = U B$ where b_{ij} are the entries of B. Therefore, by Exercise 9.4.25, $\dot{U} = A U$, where $A = C B C^{-1}$. *Q.E.D.*

(b) By induction, if $\dfrac{d^{j+1}\mathbf{u}}{dt^{j+1}} = \dfrac{d}{dt}\left(\dfrac{d^j\mathbf{u}}{dt^j}\right) = A\,\dfrac{d^j\mathbf{u}}{dt^j}$, then, differentiating the equation with respect to t, we find $d/dt\left(d^{j+1}\mathbf{u}/dt^{j+1}\right) = d/dt\left(A d^j\mathbf{u}/dt^j\right) = A d^{j+1}\mathbf{u}/dt^{j+1}$, which proves the induction step.

(c) ■

9.4.27. Prove that if $A = \begin{pmatrix} B & \mathrm{O} \\ \mathrm{O} & C \end{pmatrix}$ is a block diagonal matrix, then so is $e^{tA} = \begin{pmatrix} e^{tB} & \mathrm{O} \\ \mathrm{O} & e^{tC} \end{pmatrix}$.

Solution: Write the matrix solution to the initial value problem $\dfrac{dU}{dt} = A U$, $U(0) = \mathrm{I}$, in block form $U(t) = e^{tA} = \begin{pmatrix} V(t) & W(t) \\ Y(t) & Z(t) \end{pmatrix}$. Then the differential equation decouples into $\dfrac{dV}{dt} = B V$, $\dfrac{dW}{dt} = \mathrm{O}$, $\dfrac{dY}{dt} = \mathrm{O}$, $\dfrac{dZ}{dt} = C Z$, with initial conditions $V(0) = \mathrm{I}, W(0) = \mathrm{O}, Y(0) = \mathrm{O}, Z(0) = \mathrm{I}$. Thus, W, Y are constant, while V, Z satisfy the initial value problem for the matrix exponential, and so $V(t) = e^{tB}, W(t) = \mathrm{O}, Y(t) = \mathrm{O}, Z(t) = e^{tC}$. *Q.E.D.*

♢ 9.4.28. (a) Prove that if $J_{0,n}$ is an $n \times n$ Jordan block matrix with 0 diagonal entries, cf. (8.47),

$$\text{then } e^{t J_{0,n}} = \begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \ldots & \frac{t^n}{n!} \\ 0 & 1 & t & \frac{t^2}{2} & \ldots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \ldots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & t \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$

(b) Determine the exponential of a general Jordan block matrix $J_{\lambda,n}$. *Hint*: Use Exercise 9.4.14. (c) Explain how you can use the Jordan canonical form to compute the exponential of a matrix. *Hint*: Use Exercise 9.4.27.

Solution:

(a)

$$\frac{d}{dt}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \dots & \frac{t^n}{n!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ 0 & 0 & 0 & 1 & \dots & \frac{t^{n-3}}{(n-3)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{6} & \dots & \frac{t^n}{n!} \\ 0 & 1 & t & \frac{t^2}{2} & \dots & \frac{t^{n-1}}{(n-1)!} \\ 0 & 0 & 1 & t & \dots & \frac{t^{n-2}}{(n-2)!} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & t \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

Thus, $U(t)$ satisfies the initial value problem $\dot{U} = J_{0,n} U$, $U(0) = \mathrm{I}$, that characterizes the matrix exponential, so $U(t) = e^{t J_{0,n}}$. *Q.E.D.*

(*b*) Since $J_{\lambda,n} = \lambda\,\mathrm{I} + J_{0,n}$, by Exercise 9.4.14, $e^{t J_{\lambda,n}} = e^{t\lambda}\, e^{t J_{0,n}}$, i.e., multiply all entries in the previous formula by $e^{t\lambda}$.

(*c*) According to Exercises 9.4.17, 27, if $A = S J S^{-1}$ where J is the Jordan canonical form, then $e^{tA} = S e^{tJ} S^{-1}$, and e^{tJ} is a block diagonal matrix given by the exponentials of its individual Jordan blocks, computed in part (*b*).

♢ 9.4.29. Prove that if λ is an eigenvalue of A with multiplicity k, then $e^{t\lambda}$ is an eigenvalue of e^{tA} with the same multiplicity. *Hint*: Combine the Jordan canonical form (8.49) with Exercises 9.4.17, 28.

Solution: If J is a Jordan matrix, then, by the arguments in Exercise 9.4.28, e^{tJ} is upper triangular with diagonal entries given by $e^{t\lambda}$ where λ is the eigenvalue appearing on the diagonal of the corresponding Jordan block of A. In particular, the multiplicity of λ, which is the number of times it appears on the diagonal of J, is the same as the multiplicity of $e^{t\lambda}$ for e^{tJ}. Moreover, since e^{tA} is similar to e^{tJ}, its eigenvalues are the same, and of the same multiplicities. *Q.E.D.*

♡ 9.4.30. By a (natural) *logarithm* of a matrix B we mean a matrix A such that $e^A = B$.

(a) Explain why only nonsingular matrices can have a logarithm.

(*b*) Comparing Exercises 9.4.6–8, explain why the matrix logarithm is not unique.

(*c*) Find all real logarithms of the 2×2 identity matrix $\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. *Hint*: Use Exercise 9.4.21.

Solution: (a) All matrix exponentials are nonsingular by the remark after (9.43). (*b*) Both

$A = \mathrm{O}$ and $A = \begin{pmatrix} 0 & -2\pi \\ 2\pi & 0 \end{pmatrix}$ have the identity matrix as their exponential $e^A = \mathrm{I}$. (*c*) If $e^A = \mathrm{I}$ and λ is an eigenvalue of A, then $e^\lambda = 1$, since 1 is the only eigenvalue of I. Therefore, the eigenvalues of A must be integer multiples of $2\pi\,\mathrm{i}$. Since A is real, the eigenvalues must be complex conjugate, and hence either both 0, or $\pm 2n\pi\,\mathrm{i}$ for some positive integer n. In the latter case, since the characteristic equation of A is $\lambda^2 + 4n^2\pi^2$, A must have zero trace and determinant $4n^2\pi^2$, hence $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = -4n^2\pi^2$. If A has both eigenvalues zero, it must be complete, and hence $A = \mathrm{O}$.

9.4.31. *True or false*: The solution to the non-autonomous initial value problem $\dot{\mathbf{u}} = A(t)\,\mathbf{u}$, $\mathbf{u}(0) = \mathbf{b}$, is $\mathbf{u}(t) = e^{\int_0^t A(s)\,ds}\,\mathbf{b}$.

Solution: Even though this formula is correct in the scalar case, it is false in general. Would that life were so simple!

9.4.32. Solve the following initial value problems: (*a*) $\begin{aligned} \dot u_1 &= 2u_1 - u_2, & u_1(0) &= 0, \\ \dot u_2 &= 4u_1 - 3u_2 + e^{2t}, & u_2(0) &= 0. \end{aligned}$

(*b*) $\begin{aligned} \dot u_1 &= -u_1 + 2u_2 + e^t, & u_1(1) &= 1, \\ \dot u_2 &= 2u_1 - u_2 + e^t, & u_2(1) &= 1. \end{aligned}$ (*c*) $\begin{aligned} \dot u_1 &= -u_2, & u_1(0) &= 0, \\ \dot u_2 &= 4u_1 + \cos t, & u_2(0) &= 1. \end{aligned}$

(*d*) $\begin{aligned} \dot u &= 3u + v + 1, & u(1) &= 1, \\ \dot v &= 4u + t, & v(1) &= -1. \end{aligned}$ (*e*) $\begin{aligned} \dot p &= p + q + t, & p(0) &= 0, \\ \dot q &= -p - q + t, & q(0) &= 0. \end{aligned}$

Solution:
(a) $u_1(t) = \frac{1}{3}e^t - \frac{1}{12}e^{-2t} - \frac{1}{4}e^{2t}$, $u_2(t) = \frac{1}{3}e^t - \frac{1}{3}e^{-2t}$;
(*b*) $u_1(t) = e^{t-1} - e^t + t\,e^t$, $u_2(t) = e^{t-1} - e^t + t\,e^t$;
(*c*) $u_1(t) = \frac{1}{3}\cos 2t - \frac{1}{2}\sin 2t - \frac{1}{3}\cos t$, $u_2(t) = \cos 2t + \frac{2}{3}\sin 2t - \frac{1}{3}\sin t$;
(*d*) $u(t) = \frac{13}{16}e^{4t} + \frac{3}{16} - \frac{1}{4}t$, $v(t) = \frac{13}{16}e^{4t} - \frac{29}{16} + \frac{3}{4}t$;
(*e*) $p(t) = \frac{1}{2}t^2 + \frac{1}{3}t^3$, $q(t) = \frac{1}{2}t^2 - \frac{1}{3}t^3$.

9.4.33. Solve the following initial value problems:

(a) $\begin{aligned} \dot u_1 &= -2u_2 + 2u_3, & u_1(0) &= 1, \\ \dot u_2 &= -u_1 + u_2 - 2u_3 + t, & u_2(0) &= 0, \\ \dot u_3 &= -3u_1 + u_2 - 2u_3 + 1, & u_3(0) &= 0. \end{aligned}$ (*b*) $\begin{aligned} \dot u_1 &= u_1 - 2u_2, & u_1(0) &= -1, \\ \dot u_2 &= -u_2 + e^{-t}, & u_2(0) &= 0, \\ \dot u_3 &= 4u_1 - 4u_2 - u_3, & u_3(0) &= -1. \end{aligned}$

Solution:
(a) $u_1(t) = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{2} - \frac{1}{2}t$, $u_2(t) = 2e^{-t} - \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t - \frac{3}{2} + \frac{3}{2}t$, $u_3(t) = 2e^{-t} - \frac{1}{4}\cos 2t - \frac{3}{4}\sin 2t - \frac{7}{4} + \frac{3}{2}t$;
(*b*) $u_1(t) = -\frac{3}{2}e^t + \frac{1}{2}e^{-t} + te^{-t}$, $u_2(t) = te^{-t}$, $u_3(t) = -3e^t + 2e^{-t} + 2te^{-t}$.

9.4.34. (*a*) Write down an integral formula for the solution to the initial value problem $\dfrac{d\mathbf{u}}{dt} = A\,\mathbf{u} + \mathbf{b}$, $\mathbf{u}(0) = \mathbf{0}$, where $\mathbf{b}$ is a *constant* vector.
(*b*) Suppose $\mathbf{b} \in \operatorname{rng} A$. Do you recover the solution you found in Exercise 9.1.31?

Solution:
(a) $\mathbf{u}(t) = \displaystyle\int_0^t e^{(t-s)A}\,\mathbf{b}\,ds$.

(*b*) Yes, since if $\mathbf{b} = A\mathbf{c}$, then the integral can be evaluated as

$$\mathbf{u}(t) = \int_0^t e^{(t-s)A} A\mathbf{c}\,ds = -e^{(t-s)A}\mathbf{c}\,\Big|_{s=0}^{t} = e^{tA}\mathbf{c} - \mathbf{c} = \mathbf{v}(t) - \mathbf{u}^\star,$$

where $\mathbf{v}(t) = e^{tA}\mathbf{c}$ solves the homogeneous system $\dot{\mathbf{v}} = A\mathbf{v}$, while $\mathbf{u}^\star = \mathbf{c}$ is the equilibrium solution.

9.4.35. Suppose that λ is *not* an eigenvalue of A. Show that the inhomogeneous system $\dot{\mathbf{u}} = A\mathbf{u} + e^{\lambda t}\mathbf{v}$ has a solution of the form $\mathbf{u}^\star(t) = e^{\lambda t}\mathbf{w}$, where $\mathbf{w}$ is a constant vector. What is the general solution?

Solution: Set $\mathbf{w} = (A - \lambda\,\mathrm{I})^{-1}\mathbf{v}$. Then $\mathbf{u}^\star(t) = e^{\lambda t}\mathbf{w}$ is a solution. The general solution is $\mathbf{u}(t) = e^{\lambda t}\mathbf{w} + \mathbf{z}(t) = e^{\lambda t}\mathbf{w} + e^{tA}\mathbf{b}$, where $\mathbf{b}$ is any vector.

9.4.36. Find the one-parameter groups generated by the following matrices and interpret geometrically: What are the trajectories? What are the fixed points?

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, (*b*) $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$, (*e*) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution:

(a) $\begin{pmatrix} e^{2t} & 0 \\ 0 & 1 \end{pmatrix}$ — scalings in the x direction, which expand when $t > 0$ and contract when $t < 0$. The trajectories are half-lines parallel to the x axis. Points on the y axis are left fixed.

(*b*) $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ — shear transformations in the y direction. The trajectories are lines parallel to the y axis. Points on the y axis are fixed.

(*c*) $\begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix}$ — rotations around the origin, starting in a clockwise direction for $t > 0$. The trajectories are the circles $x^2 + y^2 = c$. The origin is fixed.

(*d*) $\begin{pmatrix} \cos 2t & -\sin 2t \\ 2\sin 2t & 2\cos 2t \end{pmatrix}$ — elliptical rotations around the origin. The trajectories are the ellipses $x^2 + \frac{1}{4}y^2 = c$. The origin is fixed.

(*e*) $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ — hyperbolic rotations. These are area-preserving scalings: for $t > 0$, expanding in the direction $x = y$ and contracting by the reciprocal factor in the direction $x = -y$; the reverse holds for $t < 0$. The trajectories are the semi-hyperbolas $x^2 - y^2 = c$ and the four rays $x = \pm y$. The origin is fixed.

9.4.37. Write down the one-parameter groups generated by the following matrices and interpret. What are the trajectories? What are the fixed points?

(a) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (*b*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (*c*) $\begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$, (*d*) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, (*e*) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Solution:

(a) $\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — scalings by a factor $\lambda = e^t$ in the y direction and $\lambda^2 = e^{2t}$ in the x direction. The trajectories are the semi-parabolas $x = cy^2, z = d$ for c, d constant, and the half-lines $x \neq 0, y = 0, z = d$ and $x = 0, y \neq 0, z = d$. Points on the z axis are left fixed.

(b) $\begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ — shear transformations in the x direction, with magnitude proportional to the z coordinate. The trajectories are lines parallel to the x axis. Points on the xy plane are fixed.

(c) $\begin{pmatrix} \cos 2t & 0 & -\sin 2t \\ 0 & 1 & 0 \\ \sin 2t & 0 & \cos 2t \end{pmatrix}$ — rotations around the y axis. The trajectories are the circles $x^2 + z^2 = c, y = d$. Points on the y axis are fixed.

(d) $\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & e^t \end{pmatrix}$ — spiral motions around the z axis. The trajectories are the positive and negative z axes, circles in the xy plane, and exponential cylindrical spirals (helices) winding around the z axis. The only fixed point is the origin.

(e) $\begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}$ — hyperbolic rotations in the xz plane, cf. Exercise 9.4.36(e). The trajectories are the semi-hyperbolas $x^2 - z^2 = c, \; y = d$, and the rays $x = \pm z, \; y = d$. The points on the y axis are fixed.

9.4.38. (a) Find the one-parameter group of rotations generated by the skew-symmetric matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. (b) As noted above, e^{tA} represents a family of rotations around a fixed axis in $\mathbb{R}^3$. What is the axis?

Solution:

(a) $$\begin{pmatrix} \frac{1}{3} + \frac{2}{3}\cos\sqrt{3}\,t & \frac{1}{3} - \frac{1}{3}\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t & -\frac{1}{3} + \frac{1}{3}\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t \\ \frac{1}{3} - \frac{1}{3}\cos\sqrt{3}\,t - \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t & \frac{1}{3} + \frac{2}{3}\cos\sqrt{3}\,t & -\frac{1}{3} + \frac{1}{3}\cos\sqrt{3}\,t - \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t \\ -\frac{1}{3} + \frac{1}{3}\cos\sqrt{3}\,t - \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t & -\frac{1}{3} + \frac{1}{3}\cos\sqrt{3}\,t + \frac{1}{\sqrt{3}}\sin\sqrt{3}\,t & \frac{1}{3} + \frac{2}{3}\cos\sqrt{3}\,t \end{pmatrix}.$$

(b) The axis is the null eigenvector: $(1, 1, -1)^T$.

♡ 9.4.39. Let $\mathbf{0} \neq \mathbf{v} \in \mathbb{R}^3$. (a) Show that the cross product $L_{\mathbf{v}}[\mathbf{x}] = \mathbf{v} \times \mathbf{x}$ defines a linear transformation on $\mathbb{R}^3$. (b) Find the 3×3 matrix representative $A_{\mathbf{v}}$ of $L_{\mathbf{v}}$ and show that it is skew-symmetric. (c) Show that every non-zero skew-symmetric 3×3 matrix defines such a cross product map. (d) Show that $\ker A_{\mathbf{v}}$ is spanned by $\mathbf{v}$. (e) Justify the fact that the matrix exponentials $e^{tA_{\mathbf{v}}}$ are rotations around the axis $\mathbf{v}$. Thus, the cross product with a vector serves as the infinitesimal generator of the one-parameter group of rotations around $\mathbf{v}$.

Solution:

(a) Given $c, d \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, we have
$$L_{\mathbf{v}}[c\mathbf{x} + d\mathbf{y}] = \mathbf{v} \times (c\mathbf{x} + d\mathbf{y}) = c\mathbf{v} \times \mathbf{x} + d\mathbf{v} \times \mathbf{y} = cL_{\mathbf{v}}[\mathbf{x}] + dL_{\mathbf{v}}[\mathbf{y}],$$
proving linearity.

(b) If $\mathbf{v} = (a, b, c)^T$, then $A_{\mathbf{v}} = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix} = -A_{\mathbf{v}}^T$.

(c) Since $A_{\mathbf{v}}\mathbf{v} = \mathbf{0}$, also $e^{tA_{\mathbf{v}}}\mathbf{v} = \mathbf{v}$, and hence the rotations fix $\mathbf{v}$.

(d) If $\mathbf{v} = r\,\mathbf{e}_3$, then $A_{r\,\mathbf{e}_3} = \begin{pmatrix} 0 & -r & 0 \\ r & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and hence $e^{tA_{r\,\mathbf{e}_3}} = \begin{pmatrix} \cos rt & -\sin rt & 0 \\ \sin rt & \cos rt & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which represent rotations around the z axis. more generally, given $\mathbf{v}$ with $r = \|\mathbf{v}\|$, let

Q be any rotation matgrix such that $Q\mathbf{v} = r\,\mathbf{e}_3$. Then $Q(\mathbf{v}\times\mathbf{x}) = (Q\mathbf{v})\times(Q\mathbf{x})$. Thus, if $\dot{\mathbf{x}} = \mathbf{v}\times\mathbf{x}$ and we set $\mathbf{y} = Q\mathbf{x}$ then $\dot{\mathbf{y}} = r\,\mathbf{e}_3\times\mathbf{y}$. Thus, the solutions $\mathbf{x}(t)$ are obtained by rotating the solutions $\mathbf{y}(t)$ and so are given by rotations around the axis $\mathbf{v}$. ■

♡ 9.4.40. Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. (*a*) Show that the solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ represents a rotation of $\mathbb{R}^3$ around the z axis. What is the trajectory of a point $\mathbf{x}_0$? (*b*) Show that the solution to the inhomogeneous system $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}$ represents a screw motion of $\mathbb{R}^3$ around the z axis. What is the trajectory of a point $\mathbf{x}_0$? (*c*) More generally, given $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^3$, show that the solution to $\dot{\mathbf{x}} = \mathbf{a}\times\mathbf{x} + \mathbf{a}$ represents a family of screw motions along the axis $\mathbf{a}$.

Solution:

(*a*) The solution is $\mathbf{x}(t) = \begin{pmatrix} x_0\cos t - y_0\sin t \\ x_0\sin t + y_0\cos t \\ z_0 \end{pmatrix}$ and so the trajectory of the point $(\,x_0, y_0, z_0\,)^T$ is the circle of radius $r_0 = \sqrt{x_0^2 + y_0^2}$ at height z_0 cenetered on the z axis. The points on the z axis, with $r_0 = 0$, are fixed.

(*b*) For the inhomogeneous system, the solution is $\mathbf{x}(t) = \begin{pmatrix} x_0\cos t - y_0\sin t \\ x_0\sin t + y_0\cos t \\ z_0 + t \end{pmatrix}$, which is a screw motion. If $r_0 = 0$, the trajectory is the z axis; otherwise it is a cylindrical helices of radius r_0 spiraling up the z axis.

(*c*) The solution to the linear system $\dot{\mathbf{x}} = \mathbf{a}\times\mathbf{x}$ is $\mathbf{x}(t) = R_t\mathbf{x}_0$ where R_t is a rotation thbrough angle $t\|\,\mathbf{a}\,\|$ around the axis $\mathbf{a}$. The solution to the inhomogeneous system is the screw motion $\mathbf{x}(t) = R_t\mathbf{x}_0 + t\,\mathbf{a}$.

♡ 9.4.41. Given a unit vector $\|\,\mathbf{u}\,\| = 1$ in $\mathbb{R}^3$, let $A = A_{\mathbf{u}}$ be the corresponding skew-symmetric 3×3 matrix that satisfies $A\mathbf{x} = \mathbf{u}\times\mathbf{x}$, as in Exercise 9.4.39. (a) Prove the *Euler–Rodrigues formula* $e^{tA} = \mathrm{I} + (\sin t)A + (1-\cos t)\,A^2$. *Hint*: Use the matrix exponential series (9.45). (*b*) Show that $e^{tA} = \mathrm{I}$ if and only if t is an integer multiple of 2π. (*c*) Generalize parts (*a*) and (*b*) to a non-unit vector $\mathbf{v} \neq \mathbf{0}$.

Solution:

(*a*) Since $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$, we have $A^2 = \begin{pmatrix} -b^2-c^2 & ab & ac \\ ab & -a^2-c^2 & bc \\ ac & bc & -a^2-b^2 \end{pmatrix}$ while $A^3 = -(a^2+b^2+c^2)U = -U$. Therefore, by induction, $U^{2m+1} = (-1)^m U$ and $U^{2m} = (-1)^{m-1}U^2$ for $m \geq 1$. Thus

$$e^{tU} = \sum_{n=0}^{\infty}\frac{t^n}{n!}\,U^n = \mathrm{I} + \sum_{m=0}^{\infty}(-1)^m\frac{t^{2m+1}}{(2m+1)!}\,U - \sum_{m=1}^{\infty}(-1)^m\frac{t^{2m}}{(2m)!}\,U^2.$$

The power series are, respectively, those of $\sin t$ and $\cos t - 1$ (since the constant term doesn't appear), proving the formula.

(*b*) Since $\mathbf{u} \neq \mathbf{0}$, the matrices I, A, A^2 are linearly independent and hence, by the Euler–Rodrigues formula, $e^{tA} = \mathrm{I}$ if and only if $\cos t = 1, \sin t = 0$, so t must be an integer multiple of 2π.

(*c*) If $\mathbf{v} = r\,\mathbf{u}$ with $r = \|\,\mathbf{v}\,\|$, then

$$e^{tA_{\mathbf{v}}} = e^{trA_{\mathbf{u}}} = \mathrm{I} + (\sin tr)A_{\mathbf{u}} + (1-\cos tr)\,A_{\mathbf{u}}^2 = \mathrm{I} + \frac{\sin t\,\|\,\mathbf{v}\,\|}{\|\,\mathbf{v}\,\|}\,A_{\mathbf{v}} + \left(1 - \frac{\cos t\,\|\,\mathbf{v}\,\|}{\|\,\mathbf{v}\,\|^2}\right)A_{\mathbf{v}}^2,$$

which equals the identity matrix if and only if $t = 2k\pi\,\|\,\mathbf{v}\,\|$ for some integer k.

9.4.42. Choose two of the groups in Exercise 9.4.36 or 9.4.37, and determine whether or not they commute by looking at their infinitesimal generators. Then verify your conclusion by directly computing the commutator of the corresponding matrix exponentials.

Solution: ■

◇ 9.4.43. Let A and B be $n \times n$ matrices. Prove that (*a*) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$; (*b*) the commutator matrix $C = [\,A, B\,] = AB - BA$ has zero trace: $\operatorname{tr} C = 0$.

Solution:

(*a*) The diagonal entries of AB are $\displaystyle\sum_{j=1}^{n} a_{ij}\, b_{ji}$, so $\operatorname{tr}(AB) = \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, b_{ji}$; the diagonal entries of BA are $\displaystyle\sum_{j=1}^{n} b_{ij}\, a_{ji}$, so $\operatorname{tr}(BA) = \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\, a_{ji}$. These double summations are clearly equal.

(*b*) $\operatorname{tr} C = \operatorname{tr}(AB - BA) = \operatorname{tr} AB - \operatorname{tr} BA = 0$ by part (*a*).

9.4.44. (*a*) Prove that the commutator of two upper triangular matrices is upper triangular.
(*b*) Prove that the commutator of two skew symmetric matrices is skew symmetric.
(*c*) Is the commutator of two symmetric matrices symmetric?

Solution:

(*a*) If U, V are upper triangular, so are UV and VU and hence so is $[\,U, V\,] = UV - VU$.

(*b*) If $A^T = -A$, $B^T = -B$ then

$$[\,A, B\,]^T = (AB - BA)^T = B^T A^T - A^T B^T = BA - AB = -[\,A, B\,].$$

(*c*) No.

◇ 9.4.45. Prove the *Jacobi identity* $[\,[\,A, B\,], C\,] + [\,[\,C, A\,], B\,] + [\,[\,B, C\,], A\,] = \mathrm{O}$ is valid for any three $n \times n$ matrices.

Solution:

(*a*)

$$\begin{aligned}
\big[\,[\,A, B\,], C\,\big] &= (AB - BA)C - C(AB - BA) = ABC - BAC - CAB + CBA,\\
\big[\,[\,C, A\,], B\,\big] &= (CA - AC)B - B(CA - AB) = CAB - ACB - BCA + BAC,\\
\big[\,[\,B, C\,], A\,\big] &= (BC - CB)A - A(BC - CB) = BCA - CBA - ABC + ACB,
\end{aligned}$$

9.4.46. Let A be an $n \times n$ matrix whose last row has all zero entries. Prove that the last row of e^{tA} is $\mathbf{e}_n^T = (\,0, \ldots, 0, 1\,)$.

Solution: In the matrix system $\dfrac{dU}{dt} = AU$, the equations in the last row are $\dfrac{du_{nj}}{dt} = 0$ for $j = 1, \ldots, n$, and hence the last row of $U(t)$ is constant. In particular, for the exponential matrix solution $U(t) = e^{tA}$ the last row must equal the last row of the identity matrix $U(0) = \mathrm{I}$, which is $\mathbf{e}_n^T$.

9.4.47. Let $A = \begin{pmatrix} B & \mathbf{c} \\ \mathbf{0} & 0 \end{pmatrix}$ be in block form, where B is an $n \times n$ matrix, $\mathbf{c} \in \mathbb{R}^n$, while $\mathbf{0}$ denotes the zero row vector with n entries. Show that its matrix exponential is also in block form $e^{tA} = \begin{pmatrix} e^{tB} & \mathbf{f}(t) \\ \mathbf{0} & 1 \end{pmatrix}$. Can you find a formula for $\mathbf{f}(t)$?

Solution: Write the matrix solution as $U(t) = \begin{pmatrix} V(t) & \mathbf{f}(t) \\ \mathbf{g}(t) & w(t) \end{pmatrix}$, where $\mathbf{f}(t)$ is a column vector,

$\mathbf{g}(t)$ a row vector, and $w(t)$ is a scalar function. Then the matrix system $\dfrac{dU}{dt} = AU$ decouples into $\dfrac{dV}{dt} = BV$, $\dfrac{d\mathbf{f}}{dt} = B\mathbf{f} + \mathbf{c}\,w$, $\dfrac{d\mathbf{g}}{dt} = \mathbf{0}$, $\dfrac{dw}{dt} = 0$, with initial conditions $V(0) = \mathrm{I}$, $\mathbf{f}(0) = \mathrm{O}$, $\mathbf{g}(0) = \mathrm{O}$, $w(0) = 1$. Thus, $\mathbf{g} \equiv \mathbf{0}$, $w \equiv 1$, are constant, $V(t) = e^{tB}$. The equation for $\mathbf{f}(t)$ becomes $\dfrac{d\mathbf{f}}{dt} = B\mathbf{f} + \mathbf{c}$, $\mathbf{f}(0) = \mathbf{0}$, and the solution is given in Exercise 9.4.34. *Q.E.D.*

◇ 9.4.48. According to Exercise 7.3.9, any $(n+1)\times(n+1)$ matrix of the block form $\begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 1 \end{pmatrix}$ in which A is an $n \times n$ matrix and $\mathbf{b} \in \mathbb{R}^n$ can be identified with the affine transformation $F[\mathbf{x}] = A\mathbf{x} + \mathbf{a}$ on $\mathbb{R}^n$. Exercise 9.4.47 shows that every matrix in the one-parameter group e^{tB} generated by $B = \begin{pmatrix} A & \mathbf{b} \\ \mathbf{0} & 0 \end{pmatrix}$ has such a form, and hence we can identify e^{tB} as a family of affine maps on $\mathbb{R}^n$. Describe the affine transformations of $\mathbb{R}^2$ generated by the following matrices:

(a) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (b) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, (c) $\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution: (a) $\begin{pmatrix} x+t \\ y \end{pmatrix}$: translations in x direction. (b) $\begin{pmatrix} e^t x \\ e^{-2t} y \end{pmatrix}$: scaling in x and y directions by respective factors $\lambda = e^t, \lambda^{-2} = e^{-2t}$. (c) $\begin{pmatrix} (x+1)\cos t - y \sin t - 1 \\ (x+1)\sin t + y\cos t \end{pmatrix}$: rotations around the point $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$. (d) $\begin{pmatrix} e^t (x+1) - 1 \\ e^{-t}(y+2) - 2 \end{pmatrix}$: scaling in x and y directions by reciprocal factors centered at the point $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

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vib

9.5. Dynamics of Structures.

9.5.1. A 6 kilogram mass is connected to a spring with stiffness 21 kg/sec^2. Determine the frequency of vibration in Hertz (cycles per second).

Solution: The vibrational frequency is $\omega = \sqrt{21/6} \approx 1.87083$, and so the number of Hertz is $\omega/(2\pi) \approx .297752$.

9.5.2. The lowest audible frequency is about 20 Hertz = 20 cycles per second. How small a mass would need to be connected to a unit spring to produce a fast enough vibration to be audible? (As always, we assume the spring has negligible mass, which is probably not so reasonable in this
situation.)

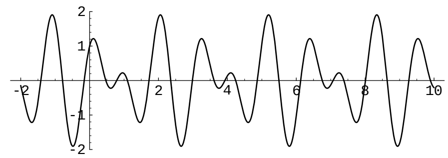
Solution: We need $\dfrac{\omega}{2\pi} = \dfrac{1}{2\pi\sqrt{m}} = 20$ and so $m = \dfrac{1}{1600\,\pi^2} \approx .0000633257$.

9.5.3. Graph the following functions. Which are periodic? quasi-periodic? If periodic, what is the (minimal) period? (a) $\sin 4t + \cos 6t$, (b) $1 + \sin \pi t$, (c) $\cos \frac{1}{2}\pi t + \cos \frac{1}{3}\pi t$,

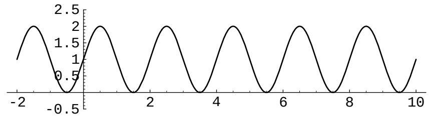
(d) $\cos t + \cos \pi t$, (e) $\sin \frac{1}{4} t + \sin \frac{1}{5} t + \sin \frac{1}{6} t$, ($f$) $\cos t + \cos \sqrt{2}\, t + \cos 2t$, ($g$) $\sin t \ \sin 3t$.

Solution:

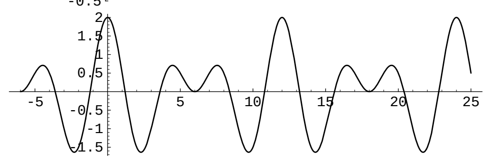
(a) Periodic of period π;



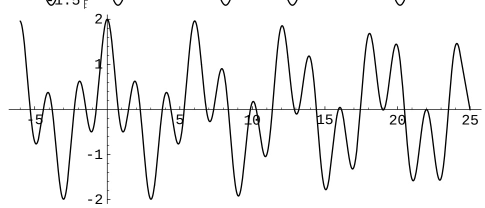
(b) Periodic of period 2;



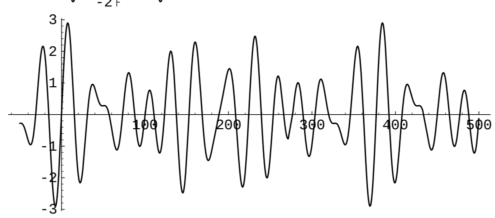
(c) Periodic of period 12;



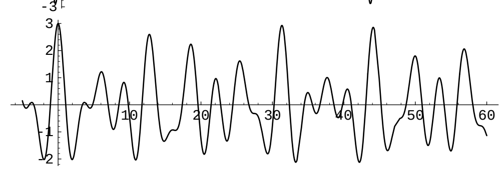
(d) Quasi-periodic;



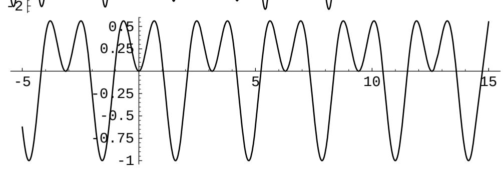
(e) Periodic of period 120π;



(f) Quasi-periodic;



(g) $\sin t \ \sin 3t = \cos 2t - \cos 4t$, and so is periodic of period π;



9.5.4. What is the minimal period of a function of the form $\cos \dfrac{p}{q}\, t + \cos \dfrac{r}{s}\, t$, assuming that each fraction is in lowest terms, i.e., its numerator and denominator have no common factors.

Solution: The minimal period is $\dfrac{\pi \ell}{2^{k-1}}$, where ℓ is the least common multiple of q and s, while 2^k is the largest power of 2 appearing in both p and r.

9.5.5. (a) Determine the natural frequencies of the Newtonian system $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \mathbf{u} = \mathbf{0}$. ($b$) What is the dimension of the space of solutions? Explain your answer. (c) Write out the general solution. (d) For which initial conditions is the resulting motion (i) periodic? (ii) quasi-periodic? (iii) both? (iv) neither? Justify your answer.

Solution: (a) $\sqrt{2}\,, \sqrt{7}\,$; (b) 4 — each eigenvalue gives two linearly independent solutions;

(c) $\mathbf{u}(t) = r_1 \cos(\sqrt{2}\,t - \delta_1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + r_2 \cos(\sqrt{7}\,t - \delta_2) \begin{pmatrix} -1 \\ 2 \end{pmatrix}$; (d) The solution is periodic if only one frequency is excited, i.e., $r_1 = 0$ or $r_2 = 0$; all other solutions are quasiperiodic.

9.5.6. Answer Exercise 9.5.5 for the system $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 73 & 36 \\ 36 & 52 \end{pmatrix} \mathbf{u} = \mathbf{0}$.

Solution: (a) $5, 10$; (b) 4 — each eigenvalue gives two linearly independent solutions;
(c) $\mathbf{u}(t) = r_1 \cos(5t - \delta_1) \begin{pmatrix} -3 \\ 4 \end{pmatrix} + r_2 \cos(10t - \delta_2) \begin{pmatrix} 4 \\ 3 \end{pmatrix}$; (d) All solutions are periodic; when r_1 (e) $ne0$, the period is $\frac{2}{5}\pi$, while when $r_1 = 0$ the period is $\frac{1}{5}\pi$.

9.5.7. Find the general solution to the following second order systems:

(a) $\dfrac{d^2u}{dt^2} = -3u + 2v, \quad \dfrac{d^2v}{dt^2} = 2u - 3v.$ (b) $\dfrac{d^2u}{dt^2} = -11u - 2v, \quad \dfrac{d^2v}{dt^2} = -2u - 14v.$

(c) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \mathbf{u} = \mathbf{0}$, (d) $\dfrac{d^2\mathbf{u}}{dt^2} = \begin{pmatrix} -6 & 4 & -1 \\ 4 & -6 & 1 \\ -1 & 1 & -11 \end{pmatrix} \mathbf{u}$.

Solution:

(a) $u(t) = r_1 \cos(t - \delta_1) + r_2 \cos(\sqrt{5}\,t - \delta_2), \quad v(t) = r_1 \cos(t - \delta_1) - r_2 \cos(\sqrt{5}\,t - \delta_2)$;

(b) $u(t) = r_1 \cos(\sqrt{10}\,t - \delta_1) - 2 r_2 \cos(\sqrt{15}\,t - \delta_2)$,
$v(t) = 2 r_1 \cos(\sqrt{10}\,t - \delta_1) + r_2 \cos(\sqrt{15}\,t - \delta_2)$;

(c) $\mathbf{u}(t) = (\, r_1 \cos(t - \delta_1), r_2 \cos(2t - \delta_2), r_3 \cos(3t - \delta_1)\,)^T$;

(d) $\mathbf{u}(t) = r_1 \cos(\sqrt{2}\,t - \delta_1) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + r_2 \cos(3t - \delta_2) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + r_3 \cos(\sqrt{12}\,t - \delta_3) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$.

9.5.8. Show that a single mass that is connected to both the top and bottom supports by two springs of stiffnesses c_1, c_2 will vibrate in the same manner as if it were connected to only one support by a spring with the combined stiffness $c = c_1 + c_2$.

Solution: The system has stiffness matrix $K = (\,1 \ \ -1\,) \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (c_1 + c_2)$ and so the dynamical equation is $m\,\ddot{u} + (c_1 + c_2)\,u = 0$, which is the same as a mass connected to a single spring with stiffness $c = c_1 + c_2$.

9.5.9. Two masses are connected by three springs to top and bottom supports. Can you find a collection of spring constants c_1, c_2, c_3 such that all vibrations are periodic?

Solution: Yes. For example, $c_1 = 16$, $c_1 = 36$, $c_1 = 37$, leads to $K = \begin{pmatrix} 52 & -36 \\ -36 & 73 \end{pmatrix}$ with eigenvalues $\lambda_1 = 25, \lambda_2 = 100$, and hence natural frequencies $\omega_1 = 5, \omega_2 = 10$. Since ω_2 is a rational multiple of ω_1, every solution is periodic with period $\frac{2}{5}\pi$ or $\frac{1}{5}\pi$. Further examples can be constructed by solving the matrix equation $K = \begin{pmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{pmatrix} = Q^T \Lambda Q$ for c_1, c_2, c_3, where Λ is a diagonal matrix with entries $\omega^2, r^2\omega^2$ where $r \in \mathbb{Q}$ is a rational number, and Q is a suitable orthogonal matrix, making sure that c_1, c_2, c_3 are all positive.

♠ 9.5.10. Suppose the bottom support in the mass-spring chain in Example 9.36 is removed. (a) Do you predict that the vibration rate will (i) speed up, (ii) slow down, or (iii) stay the same? (b) Verify your prediction by computing the new vibrational frequencies. (c) Suppose the middle mass is displaced by a unit amount and then let go. Compute and graph the solutions in both situations. Discuss what you observe.

Solution: (a) The vibrations slow down. (b) The vibrational frequencies are $\omega_1 = .44504$, $\omega_2 = 1.24698$, $\omega_3 = 1.80194$, each of which is slightly larger than the fixed end case, which are $\omega_1 = \sqrt{2-\sqrt{2}} = .76537$, $\omega_2 = \sqrt{2} = 1.41421$, $\omega_3 = \sqrt{2+\sqrt{2}} = 1.84776$. (c) ■

♡ 9.5.11. Find the vibrational frequencies for a mass–spring chain with n identical masses, connected by $n+1$ identical springs to both top and bottom supports. Is there any sort of limiting behavior as $n \to \infty$? *Hint*: See Exercise 8.2.47.

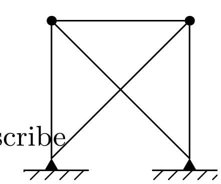
Solution: Let c be the commons spring stiffness. The stiffness matrix K is tridiagonal with all diagonal entries equal to $2c$ and all sub- and super-diagonal entries equal to $-c$. Thus, by Exercise 8.2.47, the vibrational frequencies are $\sqrt{2c\left(1-\cos\frac{k\pi}{n+1}\right)} = 2\sqrt{c}\,\sin\frac{k\pi}{2(n+1)}$ for $k = 1, \ldots, n$. As $n \to \infty$, the frequencies form a denser and denser set of points on the graph of $2\sqrt{c}\,\sin\theta$ for $0 \le \theta \le \frac{1}{2}\pi$.

♣ 9.5.12. Suppose you are given n different springs. In which order should you connect them to unit masses so that the mass-spring chain vibrates the fastest? Does your answer depend upon the relative sizes of the spring constants? Does it depend upon whether the bottom mass is attached to a support or left hanging free? First try the case of three springs with spring stiffnesses $c_1 = 1, c_2 = 2, c_3 = 3$. Then try varying the stiffnesses. Finally, predict what will happen with 4 or 5 springs, and see if you can make a conjecture in the general case.

Solution: We take "fastest" to mean that the slowest vibrational frequency is as large as possible. Keep in mind that, for a chain between two fixed supports, completely reversing the order of the springs does not change the frequencies. For the indicated springs connecting 2 masses to fixed supports, the order $2, 1, 3$ or its reverse, $3, 1, 2$ is the fastest, with frequencies $2.14896, 1.54336$. For the order $1, 2, 3$, the frequencies are 2.49721, 1.32813, while for $1, 3, 2$ the lowest frequency is the slowest, at $2.74616, 1.20773$. Note that as the lower frequency slows down, the higher one speeds up. In all cases, having the weakest spring in the middle leads to the fastest overall vibrations.

When the bottom mass is unattached, for the fastes vibration, as measured by the minimal vibrational frequency, the springs should be connected in order, from stiffest to weakest, with the strongest attached to the support. For fixed supports, it appears that if $c_1 > c_2 > \cdots > c_n$, it appears that the fastest order is $c_n, c_{n-3}, c_{n-5}, \ldots, c_3, c_1, c_2, c_4, \ldots, c_{n-1}$ if n is odd and $c_n, c_{n-1}, c_{n-4}, c_{n-6}, \ldots, c_4, c_2, c_1, c_3, \ldots, c_{n-5}, c_{n-3}, c_{n-2}$ if n is even. A challenge is to find proofs in either case.

♣ 9.5.13. Suppose the illustrated planar structure has unit masses at the nodes and the bars are all of unit stiffness. (a) Write down the system of differential equations that describes the dynamical vibrations of the structure. (b) How many independent modes of vibration are there? (c) Find numerical values for the vibrational frequencies. (d) Describe what happens when the structure vibrates in each of the normal modes. (e) Suppose the left-hand mass is displaced a unit horizontal distance. Determine the subsequent motion.



Solution: (a) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 0 & 0 \\ -1 & 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{pmatrix} \mathbf{u} = \mathbf{0}$, where $\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ v_1(t) \\ u_2(t) \\ v_2(t) \end{pmatrix}$ are the horizontal and vertical dispalcements of the two free nodes. (b) 4; (c) $\omega_1 = \sqrt{1 - \frac{1}{2}\sqrt{2}} = .541196$, $\omega_2 =$

$\sqrt{2-\frac{1}{2}\sqrt{2}} = 1.13705$, $\omega_3 = \sqrt{1+\frac{1}{2}\sqrt{2}} = 1.30656$, $\omega_4 = \sqrt{2+\frac{1}{2}\sqrt{2}} = 1.64533$; (*d*) the corresponding eigenvectors are $\mathbf{v}_1 = \begin{pmatrix} -1-\sqrt{2} \\ -1 \\ -1-\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -2.4142 \\ -1 \\ -2.4142 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} -1+\sqrt{2} \\ 1 \\ 1-\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} .4142 \\ 1 \\ -.4142 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -1+\sqrt{2} \\ -1 \\ -1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} .4142 \\ -1 \\ .4142 \\ 1 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} -1-\sqrt{2} \\ 1 \\ 1+\sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -2.4142 \\ 1 \\ 2.4142 \\ 1 \end{pmatrix}$. In the first mode, the vibrate in opposing directions vertically and proportionately 2.4 times as far in tandem horizontally; in the second mode, the move in opposite directions horizontally and proportionately 2.4 times as far in tandem vertically; ; in the third mode, the move in tandem horizontally and proportionately 2.4 times as far in opposite directions vertically; in the fourth mode, they move in tandem vertically, and proportionately 2.4 times as far in opposing directions horizontally. (*e*)

$$\mathbf{u}(t) = \frac{1}{4\sqrt{2}}\Big(-\cos(\omega_1 t)\,\mathbf{v}_1 + \cos(\omega_2 t)\,\mathbf{v}_2 + \cos(\omega_3 t)\,\mathbf{v}_3 - \cos(\omega_4 t)\,\mathbf{v}_4 \Big),$$

which is a quasiperiodic combination of all four normal modes.

9.5.14. When does a real first order linear system $\dot{\mathbf{u}} = A\,\mathbf{u}$ have a quasi-periodic solution? What is the smallest dimension in which this can occur?

Solution: The system has periodic solutions whenever A has a complex conjugate pair of puerly imaginary eigenvalues. Thus, a quasi-periodic solution requires two such pairs, $\pm\,i\,\omega_1$ and $\pm\,i\,\omega_2$, with the ratio ω_1/ω_2 an irrational number. The smallest dimension where this can occur is 4.

9.5.15. Find the general solution to the following systems. Distinguish between the vibrational and unstable modes. What constraints on the initial conditions ensure that the unstable modes are not excited? (a) $\dfrac{d^2u}{dt^2} = -4\,u - 2\,v$, $\dfrac{d^2v}{dt^2} = -2\,u - v$. (*b*) $\dfrac{d^2u}{dt^2} = -u - 3\,v$, $\dfrac{d^2v}{dt^2} = -3\,u - 9\,v$. (*c*) $\dfrac{d^2u}{dt^2} = -2\,u + v - 2\,w$, $\dfrac{d^2v}{dt^2} = u - v$, $\dfrac{d^2w}{dt^2} = -2\,u - 4\,w$. (*d*) $\dfrac{d^2u}{dt^2} = -u + v - 2\,w$, $\dfrac{d^2v}{dt^2} = u - v + 2\,w$, $\dfrac{d^2w}{dt^2} = -2\,u + 2\,v - 4\,w$.

Solution:

(a) $u(t) = a\,t + b + 2\,r\cos(\sqrt{5}\,t - \delta)$, $\quad v(t) = -2\,a\,t - 2\,b + r\cos(\sqrt{5}\,t - \delta)$.
The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $\dot{u}(t_0) - 2\,\dot{v}(t_0) = 0$.

(*b*) $u(t) = -3\,a\,t - 3\,b + r\cos(\sqrt{10}\,t - \delta)$, $\quad v(t) = a\,t + b + 3\,r\cos(\sqrt{10}\,t - \delta)$.
The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $-3\,\dot{u}(t_0) + \dot{v}(t_0) = 0$.

(*c*)

$$u(t) = -2\,a\,t - 2\,b - \tfrac{1-\sqrt{13}}{4}\,r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\;t - \delta_1\right) - \tfrac{1+\sqrt{13}}{4}\,r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\;t - \delta_2\right),$$

$$v(t) = -2\,a\,t - 2\,b + \tfrac{3-\sqrt{13}}{4}\,r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\;t - \delta_1\right) + \tfrac{3+\sqrt{13}}{4}\,r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\;t - \delta_2\right),$$

$$w(t) = a\,t + b + r_1\cos\left(\sqrt{\tfrac{7+\sqrt{13}}{2}}\;t - \delta_1\right) + r_2\cos\left(\sqrt{\tfrac{7-\sqrt{13}}{2}}\;t - \delta_2\right).$$

The unstable mode consists of the terms with a in them; it will not be excited if the initial conditions satisfy $-2\dot u(t_0) - 2\dot v(t_0) + \dot w(t_0) = 0$.

(d)

$$\begin{aligned} u(t) &= (a_1 - 2a_2)t + b_1 - 2b_2 + r\cos(\sqrt{6}\,t - \delta),\\ v(t) &= a_1 t + b_1 - r\cos(\sqrt{6}\,t - \delta),\\ w(t) &= a_2 t + b_2 + 2r\cos(\sqrt{6}\,t - \delta). \end{aligned}$$

The unstable modes consists of the terms with a_1 and a_2 in them; they will not be excited if the initial conditions satisfy $\dot u(t_0) + \dot v(t_0) = 0$ and $-2\dot u(t_0) + \dot w(t_0) = 0$.

9.5.16. Let $K = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$. (a) Find an orthogonal matrix Q and a diagonal matrix Λ such that $K = Q\Lambda Q^T$. (b) Is K positive definite? (c) Solve the second order system $\dfrac{d^2\mathbf{u}}{dt^2} = A\mathbf{u}$ subject to the initial conditions $\mathbf{u}(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\dfrac{d\mathbf{u}}{dt}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. (d) Is your solution periodic? If your answer is "yes", indicate the period. (e) Is the general solution to the system periodic?

Solution: (a) $Q = \begin{pmatrix} -\frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} & 0 \end{pmatrix}$, $\Lambda = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; (b) yes, because K is symmetric and has all positive eigenvalues; (c) $\mathbf{u}(t) = \left(\cos\sqrt2\,t, \dfrac{1}{\sqrt2}\sin\sqrt2\,t, \cos\sqrt2\,t\right)^T$; (d) the solution $\mathbf{u}(t)$ is periodic with period $\sqrt2\,\pi$; (e) no — since the frequencies $2, \sqrt2$ are not rational mutliples of each other, the general solution is quasi-periodic.

9.5.17. Answer Exercise 9.5.16 when $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

Solution:

(a) $Q = \begin{pmatrix} \frac{1}{\sqrt3} & -\frac{1}{\sqrt2} & \frac{1}{\sqrt6} \\ -\frac{1}{\sqrt3} & 0 & \frac{2}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt2} & \frac{1}{\sqrt6} \end{pmatrix}$, $\Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$;

(b) no — K is only positive semi-definite;

(c) $\mathbf{u}(t) = \begin{pmatrix} \frac13(t+1) + \frac23\cos\sqrt3\,t - \frac{1}{3\sqrt3}\sin\sqrt3\,t \\ \frac23(t+1) - \frac23\cos\sqrt3\,t + \frac{1}{3\sqrt3}\sin\sqrt3\,t \\ \frac13(t+1) + \frac23\cos\sqrt3\,t - \frac{1}{3\sqrt3}\sin\sqrt3\,t \end{pmatrix}$;

(d) the solution $\mathbf{u}(t)$ is unstable, and becomes unbounded as $|\,t\,| \to \infty$;

(e) no — the general solution is also unbounded.

9.5.18. Compare the solutions to the mass-spring system (9.59) with tiny spring constant $k = \varepsilon \ll 1$ to those of the completely unrestrained system (9.71). Are they close? Discuss.

Solution: The solution to the initial value problem $m\dfrac{d^2u}{dt^2} + \varepsilon u = 0$, $u(t_0) = a$, $\dot u(t_0) = b$, is

$u_\varepsilon(t) = a \cos \sqrt{\dfrac{\varepsilon}{m}}\,(t - t_0) + b \sqrt{\dfrac{m}{\varepsilon}}\, \sin \sqrt{\dfrac{\varepsilon}{m}}\,(t - t_0)$. In the limit as $\varepsilon \to 0$, using the fact that $\displaystyle\lim_{h \to 0} \frac{\sin c\,h}{h} = c$, we find $u_\varepsilon(t) \longrightarrow a + b(t - t_0)$, which is the solution to the unrestrained initial value problem $m\,\ddot u = 0,\ u(t_0) = a,\ \dot u(t_0) = b$. Thus, as the spring stiffness goes to zero, the motion converges to the unrestrained motion. However, since the former solution is periodic, while the latter moves along a straight line, the convergence is noo-uniform on all of $\mathbb{R}$ and the solutions are close only for a period of time: if you wait long enough they will diverge.

♠ 9.5.19. Find the vibrational frequencies and instabilities of the following structures, assuming they have unit masses at all the nodes. Explain in detail how each normal mode moves the structure: (*a*) the three bar planar structure in Figure 6.13, (*b*) its reinforced version in Figure 6.16, (*c*) the swing set in Figure 6.18.

Solution:

(*a*) Frequencies: $\omega_1 = \sqrt{\tfrac32 - \tfrac12\sqrt5} = .61803,\ \omega_2 = 1,\ \omega_3 = \sqrt{\tfrac32 + \tfrac12\sqrt5} = 1.618034$; stable eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} -2-\sqrt5 \\ -1 \\ -2+\sqrt5 \\ 1 \end{pmatrix},\ \mathbf{v}_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix},\ \mathbf{v}_3 = \begin{pmatrix} 2+\sqrt5 \\ 1 \\ -2-\sqrt5 \\ 1 \end{pmatrix}$; unstable eigenvector: $\mathbf{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$. In the lowest frequency mode, the nodes vibrate up and towards each other and then down and away, the horizontal motion being less pronounced than the vertical; in the next mode, the nodes vibrate up and away from each other and then down and together, along directions bisecting the angle between the two bars; in the highest frequency mode, they also vibrate up and away from each other and then down and towards, with the horizontal motion significantly more than the vertical; in the unstable mode the left node moves down and to the right, while the right hand node moves at the same rate up and to the right.

(*b*) Frequencies: $\omega_1 = .444569,\ \omega_2 = .758191,\ \omega_3 = 1.06792,\ \omega_4 = 1.757$; eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} .23727 \\ -.11794 \\ .498965 \\ .825123 \end{pmatrix},\quad \mathbf{v}_2 = \begin{pmatrix} -.122385 \\ .973375 \\ -.0286951 \\ .191675 \end{pmatrix},\quad \mathbf{v}_3 = \begin{pmatrix} .500054 \\ .185046 \\ .666846 \\ -.520597 \end{pmatrix},\quad \mathbf{v}_4 = \begin{pmatrix} .823815 \\ .0662486 \\ -.552745 \\ .10683 \end{pmatrix}.$$

In the lowest frequency mode, the left node vibrates down and to the right, while the right hand node moves at the same rate up and to the right, then both reversing diredtions; in the second mode, the nodes vibrate up and towards each other, and then down and away from each other, the horizontal motion being less pronounced than the vertical; in the next mode, the nodes vibrate towards and then away from each other along directions bisecting the angle between the two bars; in the highest frequency mode, they also vibrate up and away from each other and then down and towards, with the horizontal motion more than the vertical.

(*c*) Frequencies: $\omega_1 = \sqrt{\tfrac{2}{11}} = .426401,\ \omega_2 = \sqrt{\tfrac{2}{11}} = .426401,\ \omega_3 = \sqrt{\tfrac{21}{11} - \tfrac{3}{11}\sqrt5} = 1.13985$, $\omega_4 = \sqrt{\tfrac{20}{11}} = 1.3484,\ \omega_5 = \sqrt{\tfrac{21}{11} + \tfrac{3}{11}\sqrt5} = 1.58711$; stable eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},\quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\quad \mathbf{v}_3 = \begin{pmatrix} \frac12 - \frac{\sqrt5}{2} \\ 0 \\ 1 \\ -\frac12 + \frac{\sqrt5}{2} \\ 0 \\ 1 \end{pmatrix},\quad \mathbf{v}_4 = \begin{pmatrix} -\frac13 \\ 0 \\ -1 \\ -\frac13 \\ 0 \\ 1 \end{pmatrix},\quad \mathbf{v}_5 = \begin{pmatrix} \frac12 + \frac{\sqrt5}{2} \\ 0 \\ 1 \\ -\frac12 - \frac{\sqrt5}{2} \\ 0 \\ 1 \end{pmatrix};$$

unstable eigenvector: $\mathbf{v}_6 = \begin{pmatrix} 3 \\ 0 \\ -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$. In the two lowest frequency modes, the individual nodes vibrate horizontally and transverse to the swing; in the next lowest mode, the nodes vibrate together up and away from each other, and then down and towards each other; in the next mode, the nodes vibrate up and down in opposing motion, and towards and then away from each other; in the highest frequency mode, they also vibrate vibrate up and down in opposing motion, but in the same direction along the swing; in the unstable mode the left node moves down and in the direction of the bar, while the right hand node moves at the same rate up and in the same horizontal direction.

♡ 9.5.20. Discuss the three-dimensional motions of the triatomic molecule of Example 9.37. Are the vibrational frequencies the same as the one-dimensional model?

Solution: If the mass/spring molecule is allowed to move in space, then the vibrational modes and frequencies remain the same, while there are 14 independent solutions corresponding to the 7 modes of instability: 3 rigid translations, 3 (linearized) rotations, and 1 mechanism, which is the same as in the one-dimensional version. Thus, the general motion of the molecule in space is to vibrate quasi-periodically at frequencies $\sqrt{3}$ and 1, while simultaneously translating, rigidly rotating, and bending, all at a constant speed. (Of course, these are still just linear approximations to the full nonlinear motions, aand also do not take into account interatomic repulsive forces that prevent the two end atoms from getting too close to each other.)

♠ 9.5.21. Assuming unit masses at the nodes, find the vibrational frequencies and and describe the normal modes for the following planar structures. What initial conditions will not excite its instabilities (rigid motions and/or mechanisms)? (*a*) An equilateral triangle; (*b*) a square; (*c*) a regular hexagon.

Solution:

(*a*) There are 3 linearly independent normal modes of vibration: one of frequency $\sqrt{3}$, , in which the triangle expands and contacts, and two of frequency $\sqrt{\frac{3}{2}}$, , in which one of the edges expands and contracts while the opposite vertex moves out in the perpendicular direction while the edge is contracting, and in when it expands. (Although there are three such modes, the third is a linear combination of the other two.) There are 3 unstable null eigenmodes, corresponding to the planar rigid motions of the triangle. To avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel; thus, if $\mathbf{v}_i$ is the initial velocity of the i^{th} mode, we require $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ and $v_1^\perp + v_2^\perp + v_3^\perp = 0$ where $v_i^\perp$ denotes the angular component of thhe velocity vector with respect to the center of the triangle.

(*b*) There are 4 normal modes of vibration, all of frequency $\sqrt{2}$, in which one of the edges expands and contracts while the two vertices not on the edge stay fixed. There are 4 unstable modes: 3 rigid motions and one mechanism where two opposite corners move towards each other while the other two move away from each other. To avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel; thus, if the vertices are at $(\pm 1, \pm 1)^T$ and $\mathbf{v}_i = (v_i, w_i)^T$ is the initial velocity of the i^{th} mode, we require $v_1 + v_2 = v_3 + v_4 = w_1 + w_4 = w_2 + w_3 = 0$.

(*c*) There are 6 normal modes of vibration: one of frequency $\sqrt{3}$, in which three nonadjacent edges expand and then contact, while the other three edges simultaneously contract and then expand; two of frequency $\sqrt{\frac{5}{2}}$, in which two opposite vertices move back and forth in the perpendicular direction to the line joining them (onoly two of these three modes are linearly independent); two of frequency $\sqrt{\frac{3}{2}}$, in which two opposite vertices move back and forth towards each other (again, only two of these three modes are linearly independent); and one of frequency 1, in which the entire hexagon expands and contacts. There are 6 unstable modes: 3 rigid motions and 3 mechanisms where two opposite vertice move towards each other while the other four move away. As usual, to avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel; thus, if the vertices are at $\left(\cos\frac{1}{3}k\pi, \sin\frac{1}{3}k\pi\right)^T$, and $\mathbf{v}_i = (v_i, w_i)^T$ is the initial velocity of the i^{th} mode, we require

$$\begin{aligned} v_1+v_2+v_3+v_4+v_5+v_6 &= 0, & -\sqrt{3}\,v_1+w_1+2w_2 &= 0,\\ w_1+w_2+w_3+w_4+w_5+w_6 &= 0, & \sqrt{3}\,v_1+w_1+2w_6 &= 0,\\ \sqrt{3}\,v_5+w_5+\sqrt{3}\,v_6+w_6 &= 0, & 2w_3+\sqrt{3}\,v_4+w_4 &= 0. \end{aligned}$$

♠ 9.5.22. Answer Exercise 9.5.21 for the three-dimensional motions of a regular tetrahedron.

Solution: There are 6 linearly independent normal modes of vibration: one of frequency 2, in which the tetrahedron expands and contacts; four of frequency $\sqrt{2}$, in which one of the edges expands and contracts while the opposite vertex stays fixed; and two of frequency $\sqrt{2}$, in which two opposite edges move towards and away from each other. (There are three different pairs, but the third mode is a linear combination of the other two.) There are 6 unstable null eigenmodes, corresponding to the three-dimensional rigid motions of the tetrahedron.

To avoid exciting the instabilities, the initial velocity must be orthogonal to the kernel, and so, using the result of Exercise 6.3.13, if $\mathbf{v}_i = (u_i, v_i, w_i)^T$ is the initial velocity of the i^{th} mode, we require

$$\begin{aligned} u_1+u_2+u_3+u_4 &= 0, & -\sqrt{2}\,u_1+\sqrt{6}\,v_1-w_1-2\sqrt{2}\,u_2+w_2 &= 0,\\ v_1+v_2+v_3+v_4 &= 0, & -2v_1+\sqrt{3}\,u_2+v_2-\sqrt{3}\,u_3+v_3 &= 0,\\ w_1+w_2+w_3+w_4 &= 0, & -\sqrt{2}\,u_1-\sqrt{6}\,v_1-w_1-2\sqrt{2}\,u_3+w_3 &= 0. \end{aligned}$$

♡ 9.5.23. (*a*) Show that if a structure contains all unit masses and bars with unit stiffness, $c_i = 1$, then its frequencies of vibration are the nonzero singular values of the reduced incidence matrix. (*b*) How would you recognize when a structure is close to being unstable?

Solution: (*a*) When $C = \mathrm{I}$, then $K = A^T A$ and so the frequencies $\omega_i = \sqrt{\lambda_i}$ are the square roots of its positive eigenvalues, which, by definition, are the singular values of the reduced incidence matrix. (*b*) Thus, a stucture with one or more very small frequencies $\omega_i \ll 1$, and hence one or more very slow vibrational modes, is almost unstable in that a small perturbation can create a null eigenvalue corresponding to a low frquency mode.

9.5.24. Prove that if the initial velocity satisfies $\dot{\mathbf{u}}(t_0) = \mathbf{b} \in \operatorname{corng} A$, then the solution to the initial value problem (9.63, 69) remains bounded.

Solution: Since $\operatorname{corng} A$ is the orthogonal complement to $\ker A = \ker K$, the initial velocity is orthogonal to all modes of instability, and hence by Theorem 9.38, the solution remains bounded, vibrating around the fixed point prescribed by the initial position.

9.5.25. Find the general solution to the system (9.75) for the following matrix pairs:

(*a*) $M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $K = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$, (*b*) $M = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$, $K = \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix}$,

(*c*) $M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $K = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, (*d*) $M = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$, $K = \begin{pmatrix} 5 & -1 & -1 \\ -1 & 6 & 3 \\ -1 & 3 & 9 \end{pmatrix}$,

$(e)\ M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad K = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad (f)\ M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 1 \end{pmatrix}.$

Solution:

(a) $\mathbf{u}(t) = r_1 \cos\left(\frac{1}{\sqrt{2}}\,t - \delta_1\right)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + r_2 \cos\left(\sqrt{\tfrac{5}{3}}\,t - \delta_2\right)\begin{pmatrix} -3 \\ 1 \end{pmatrix};$

(b) $\mathbf{u}(t) = r_1 \cos\left(\frac{1}{\sqrt{3}}\,t - \delta_1\right)\begin{pmatrix} 2 \\ 3 \end{pmatrix} + r_2 \cos\left(\sqrt{\tfrac{8}{5}}\,t - \delta_2\right)\begin{pmatrix} -5 \\ 2 \end{pmatrix};$

(c) $\mathbf{u}(t) = r_1 \cos\left(\sqrt{\tfrac{3-\sqrt{3}}{2}}\,t - \delta_1\right)\begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ 1 \end{pmatrix} + r_2 \cos\left(\sqrt{\tfrac{3+\sqrt{3}}{2}}\,t - \delta_2\right)\begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ 1 \end{pmatrix};$

(d) $\mathbf{u}(t) = r_1 \cos(t - \delta_1)\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + r_2 \cos\left(\sqrt{2}\,t - \delta_2\right)\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + r_3 \cos\left(\sqrt{3}\,t - \delta_3\right)\begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix};$

(e) $\mathbf{u}(t) = r_1 \cos\left(\sqrt{\tfrac{2}{3}}\,t - \delta_1\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + r_2 \cos(2t - \delta_2)\begin{pmatrix} -1 \\ 1 \end{pmatrix};$

(f) $\mathbf{u}(t) = (at + b)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + r_1 \cos(t - \delta_1)\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + r_2 \cos\left(\sqrt{3}\,t - \delta_2\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$

9.5.26. A mass-spring chain consisting of two masses, $m_1 = 1$ and $m_2 = 2$ connected to top and bottom supports by identical springs with unit stiffness. The upper mass is displaced by a unit distance. Find the subsequent motion of the system.

Solution:

$$u_1(t) = \frac{\sqrt{3}-1}{2\sqrt{3}}\cos\sqrt{\frac{3-\sqrt{3}}{2}}\,t + \frac{\sqrt{3}+1}{2\sqrt{3}}\cos\sqrt{\frac{3+\sqrt{3}}{2}}\,t, \quad u_2(t) = \frac{1}{2\sqrt{3}}\cos\sqrt{\frac{3-\sqrt{3}}{2}}\,t - \frac{1}{2\sqrt{3}}\cos\sqrt{\frac{3+\sqrt{3}}{2}}\,t.$$

9.5.27. Answer Exercise 9.5.26 when the bottom support is removed.

Solution:

$$u_1(t) = \frac{\sqrt{17}-3}{2\sqrt{17}}\cos\frac{\sqrt{5-\sqrt{17}}}{2}\,t + \frac{\sqrt{17}+3}{2\sqrt{17}}\cos\frac{\sqrt{5+\sqrt{17}}}{2}\,t,$$
$$u_2(t) = \frac{1}{\sqrt{17}}\cos\frac{\sqrt{5-\sqrt{17}}}{2}\,t - \frac{1}{\sqrt{17}}\cos\frac{\sqrt{5+\sqrt{17}}}{2}\,t.$$

♠ 9.5.28. Suppose you have masses $m_1 = 1, m_2 = 2, m_3 = 3$ connected to top and bottom supports by identical unit springs. Does rearranging the order of the masses change the fundamental frequencies? If so, which order produces the fastest vibrations?

♣ 9.5.29. (*a*) A water molecule consists of two hydrogen atoms connected at an angle of 105° to an oxygen atom whose relative mass is 16 times that of the hydrogen atoms. If the bonds are modeled as linear unit springs, determine the fundamental frequencies and modes of vibrations. (*b*) Do the same for a carbon tetrachloride molecule, in which the chlorine atoms, with atomic weight 35, are positioned on the vertices of a regular tetrahedron and the carbon atom, with atomic weight 12, is at the center. (*c*) Finally try a benzene molecule, consisting of 6 carbon atoms arranged in a regular hexagon. In this case, every other bond is double strength because two electrons are shared. (Ignore the six extra hydrogen atoms for simplicity.)

♣ 9.5.30. So far, our mass-spring chain has only been allowed to move in the vertical direction. (*a*) Set up the system governing the planar motions of a mass–spring chain consisting two masses attached to top and bottom supports where the masses are allowed to move in the longitudinal and transverse directions. Compare the resulting vibrational frequencies with the 1-dimensional case. (*b*) Repeat the analysis when the bottom support is removed. (*c*) Can you make any conjectures concerning the planar motions of general mass-spring chains?

♣ 9.5.31. Repeat Exercise 9.5.30 for fully 3-dimensional motions of the chain.

♢ 9.5.32. Suppose M is a nonsingular matrix. Prove that λ is a generalized eigenvalue of the matrix pair K, M if and only if it is an ordinary eigenvalue of the matrix $P = M^{-1}K$. How are the eigenvectors related? How are the characteristic equations related?

Solution: $K\mathbf{v} = \lambda M\mathbf{v}$ if and only if $M^{-1}K\mathbf{v} = \lambda\mathbf{v}$, and so the eigenvectors are the same. The characteristic equations are the same up to a multiple, since

$$\det(K - \lambda M) = \det\Big[M(M^{-1}K - \lambda\, \mathrm{I}) \Big] = \det M\ \det(P - \lambda\, \mathrm{I}).$$

9.5.33. Suppose that $\mathbf{u}(t)$ is a solution to (9.75). Let $N = \sqrt{M}$ denote the positive definite square root of the mass matrix M, as defined in Exercise 8.4.26. (*a*) Prove that the "weighted" displacement vector $\widetilde{\mathbf{u}}(t) = N\,\mathbf{u}(t)$ solves $d^2\widetilde{\mathbf{u}}/dt^2 = -\widetilde{K}\,\widetilde{\mathbf{u}}$, where $\widetilde{K} = N^{-1}K\,N^{-1}$ is a symmetric, positive semi-definite matrix. (*b*) Explain in what sense this can serve as an alternative to the generalized eigenvector solution method.

Solution:

(a) First,

$$\frac{d^2\widetilde{\mathbf{u}}}{dt^2} = N\,\frac{d^2\mathbf{u}}{dt^2} = -N\,K\,\mathbf{u} = -N\,K\,N^{-1}\mathbf{u} = -\widetilde{K}\,\widetilde{\mathbf{u}}.$$

Moreover, $\widetilde{K}$ is symmetric since $\widetilde{K}^T = N^{-T}K^TN^{-T} = N^{-1}KN^{-1}$ since both N and K are symmetric. Positive definiteness follows since

$$\widetilde{\mathbf{x}}^T\,\widetilde{K}\,\widetilde{\mathbf{x}} = \widetilde{\mathbf{x}}^T\,N^{-1}K\,N^{-1}\,\widetilde{\mathbf{x}} = \mathbf{x}^T K\,\mathbf{x} > 0 \qquad \text{for all} \qquad \widetilde{\mathbf{x}} = N\,\mathbf{x} \neq \mathbf{0}.$$

(b) Each eigenvalue $\widetilde{\lambda} = \widetilde{\omega}^2$ and corresponding eigenvector $\widetilde{\mathbf{v}}$ of $\widetilde{K}$ produces two solutions $\widetilde{\mathbf{u}}(t) = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\} \widetilde{\omega}\,t\,\widetilde{\mathbf{v}}$ to the modified system $d^2\widetilde{\mathbf{u}}/dt^2 = -\widetilde{K}\,\widetilde{\mathbf{u}}$. The corresponding solutions to the original system are $\mathbf{u}(t) = N^{-1}\widetilde{\mathbf{u}}(t) = \left\{ \begin{smallmatrix} \cos \\ \sin \end{smallmatrix} \right\} \omega\,t\,\mathbf{v}$, where $\omega = \widetilde{\omega}$ and $\mathbf{v} = N^{-1}\widetilde{\mathbf{v}}$. Finally, we observe that $\mathbf{v}$ is the generalized eigenvector for the generalized eigenvalue $\lambda = \omega^2 = \widetilde{\lambda}$ of the matrix pair K, M. Indeed, $\widetilde{K}\,\widetilde{\mathbf{v}} = \widetilde{\lambda}\,\widetilde{\mathbf{v}}$ implies $K\mathbf{v} = N\,K\,N\,N^{-1}\mathbf{v} = \widetilde{\lambda}\,\mathbf{v}$.

♢ 9.5.34. Provide the details of the proof of Theorem 9.38.

Solution: ■

9.5.35. Solve the following mass-spring initial value problems, and classify as to (*i*) overdamped, (*ii*) critically damped, (*iii*) underdamped, or (*iv*) undamped:
(*a*) $\ddot{u} + 6\dot{u} + 9\,u = 0,\ u(0) = 0,\ \dot{u}(0) = 1.$ (*b*) $\ddot{u} + 2\,\dot{u} + 10\,u = 0,\ u(0) = 1,\ \dot{u}(0) = 1.$
(*c*) $\ddot{u} + 16\,u = 0,\ u(1) = 0,\ \dot{u}(1) = 1.$ (*d*) $\ddot{u} + 3\,\dot{u} + 9\,u = 0,\ u(0) = 0,\ \dot{u}(0) = 1.$
(*e*) $2\,\ddot{u} + 3\,\dot{u} + u = 0,\ u(0) = 2,\ \dot{u}(0) = 0.$ (*f*) $\ddot{u} + 6\,\dot{u} + 10\,u = 0,\ u(0) = 3,\ \dot{u}(0) = -2.$

Solution:

(a) $u(t) = t\,e^{-3t}$. Critically damped.
(*b*) $u(t) = e^{-t}\left(\cos 3t + \frac{2}{3}\sin 3t\right)$. Underdamped.
(*c*) $u(t) = \frac{1}{4}\sin 4\,(t-1)$. Undamped.
(*d*) $u(t) = \frac{2\sqrt{3}}{9}\,e^{-3t/2}\sin\frac{3\sqrt{3}}{2}\,t$. Underdamped.
(*e*) $u(t) = 4\,e^{-t/2} - 2\,e^{-t}$. Overdamped.
(*f*) $u(t) = e^{-3t}(3\cos t + 7\sin t)$. Underdamped.

9.5.36. Consider the overdamped mass-spring equation $\ddot{u} + 6\dot{u} + 5u = 0$. (a) If the mass starts out a distance 1 away from equilibrium, how large must the initial velocity be in order that it pass through equilibrium once?

Solution: The solution is $u(t) = \frac{1}{4}(v+5)e^{-t} - \frac{1}{4}(v+1)e^{-5t}$ where $v = \dot{u}(0)$ is the initial velocity. This vanishes when $e^{4t} = \dfrac{v+1}{v+5}$, which happens when $t > 0$ provided $\dfrac{v+1}{v+5} > 1$, and so the initial velocity must satisfy $v < -5$.

9.5.37. (*a*) A mass weighing 16 pounds stretches a spring 6.4 feet. Assuming no friction, determine the equation of motion and the natural frequency of vibration of the mass-spring system. Use the value $g = 32\,\text{ft/sec}^2$ for the gravitational acceleration. (*b*) The mass-spring system is placed in a jar of oil, whose frictional resistance equals the speed of the mass. Assume the spring is stretched an additional 2 feet from its equilibrium position and let go. Determine the motion of the mass. (*c*) Is the system over-damped or under-damped? Are the vibrations more rapid or less rapid than the undamped system?

Solution:

(*a*) By Hooke's Law, the spring stiffness is $k = 16/6.4 = 2.5$. The mass is $16/32 = .5$. The equations of motion are $.5\,\ddot{u} + 2.5\,u = 0$. The natural frequency is $\omega = \sqrt{5} = 2.23607$.

(*b*) The solution to the initial value problem $.5\,\ddot{u} + \dot{u} + 2.5\,u = 0 u(0) = 2,\ \dot{u}(0) = 0$, is $u(t) = e^{-t}(2\cos 2t + \sin 2t)$.

(*c*) The system is underdamped, and the vibrations are less rapid than the undamped system.

9.5.38. Suppose you convert the second order equation (9.80) into its phase plane equivalent. What are the phase portraits corresponding to (*a*) undamped, (*b*) underdamped, (*c*) critically damped, and (*d*) overdamped motion?

Solution: The undamped case corresponds to a center, the underdamped case to a stable focus, the critically damped case to a stable improper node, and the overdamped case to a stable node.

♢ 9.5.39. (*a*) Prove that, for any non-constant solution to an overdamped mass-spring system, there is at most one time where $u(t_\star) = 0$. (*b*) Is this statement also valid in the critically damped case?

Solution:

(*a*) The general solution has the form $u(t) = c_1 e^{-at} + c_2 e^{-bt}$ for some $0 < a < b$. If $c_1 = 0$, $c_2 \neq 0$, the solution does not vanish. Otherwise, $u(t) = 0$ if and only if $e^{(b-a)t} = -c_2/c_1$, which, since $e^{(b-a)t}$ is monotonic, happens for at most one t.

(*b*) Yes, since the solution is $u(t) = (c_1 + c_2 t)e^{-at}$ for some $a > 0$, which, for $c_2 \neq 0$, only vanishes when $t = -c_1/c_2$.

9.5.40. Discuss the possible behaviors of a mass moving in a frictional medium that is not attached to a spring, i.e., set $k = 0$ in (9.80).

Solution: The general solution to $m\dfrac{d^2u}{dt^2} + \beta\dfrac{du}{dt} = 0$ is $u(t) = c_1 + c_2 e^{-\beta t/m}$. Thus, the mass approaches its equilibrium position, which can be anywhere, at an exponentially fast rate.

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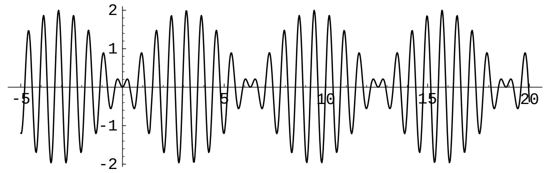
resonance

9.6. Forcing and Resonance.

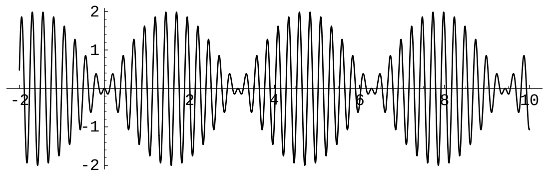
9.6.1. Graph the following functions. Describe the fast oscillation and beat frequencies: (*a*) $\cos 8t - \cos 9t$, (*b*) $\cos 26t - \cos 24t$, (*c*) $\cos 10t + \cos 9.5t$, (*d*) $\cos 5t - \sin 5.2t$.

Solution:

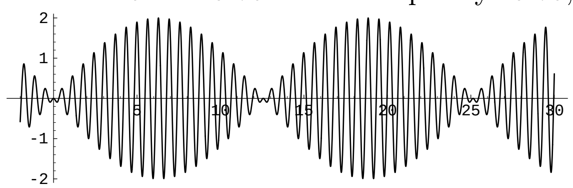
(*a*) $\cos 8t - \cos 9t = 2\sin \frac{1}{2}t \ \sin \frac{17}{2}t$. Fast frequency: $\frac{17}{2}$, beat frequency: $\frac{1}{2}$;



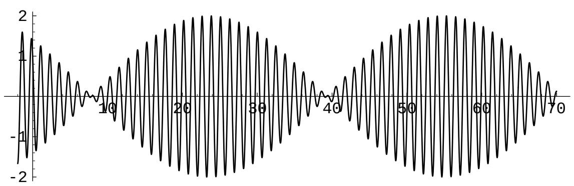
(*b*) $\cos 26t - \cos 24t = -2\sin t \ \sin 25t$. Fast frequency: 25, beat frequency: 1;



(*c*) $\cos 10t + \cos 9.5t = 2\sin .25t \ \sin 9.75t$. Fast frequency: 9.75, beat frequency: .25;



(*d*) $\cos 5t - \sin 5.2t = 2\sin\left(.1t - \frac{1}{4}\pi\right) \ \sin\left(5.1t - \frac{1}{4}\pi\right)$. Fast frequency: 5.1, beat frequency: .1;



♠ 9.6.2. Does a function of the form $u(t) = a\cos\eta t - b\cos\omega t$ still exhibit beats when $\eta \approx \omega$, but $a \neq b$? Use a computer to graph some particular cases and discuss what you observe.

Solution: Yes, the same fast oscillations and beats can be observed graphically, even though there is no elementary trigonometric identity that can give a simple explanation. ■

9.6.3. Solve the following initial value problems: (*a*) $\ddot{u} + 36u = \cos 3t$, $u(0) = 0$, $\dot{u}(0) = 0$. (*b*) $\ddot{u} + 6\dot{u} + 9u = \cos t$, $u(0) = 0$, $\dot{u}(0) = 1$. (*c*) $\ddot{u} + \dot{u} + 4u = \cos 2t$, $u(0) = 1$, $\dot{u}(0) = -1$. (*d*) $\ddot{u} + 9u = 3\sin 3t$, $u(0) = 1$, $\dot{u}(0) = -1$. (*e*) $2\ddot{u} + 3\dot{u} + u = \cos\frac{1}{2}t$, $u(0) = 3$, $\dot{u}(0) = -2$. (*f*) $3\ddot{u} + 4\dot{u} + u = \cos t$, $u(0) = 0$, $\dot{u}(0) = 0$.

Solution:

(a) $u(t) = \frac{1}{27}\cos 3t - \frac{1}{27}\cos 6t$;

(*b*) $u(t) = \frac{35}{50}te^{-3t} - \frac{4}{50}e^{-3t} + \frac{4}{50}\cos t + \frac{3}{50}\sin t$;

(*c*) $u(t) = \frac{1}{2}\sin 2t + e^{-t/2}\left(\cos\frac{\sqrt{15}}{2}t - \frac{\sqrt{15}}{5}\sin\frac{\sqrt{15}}{2}t\right)$;

(*d*) $u(t) = \cos 3t - \frac{1}{2}t\cos 3t - \frac{1}{6}\sin 3t$;

(*e*) $u(t) = \frac{1}{5}\cos\frac{1}{2}t + \frac{3}{5}\sin\frac{1}{2}t + \frac{9}{5}e^{-t} + e^{-t/2}$;

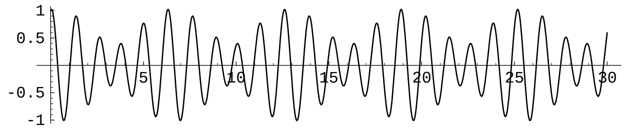
(*f*) $u(t) = -\frac{1}{10}\cos t + \frac{1}{5}\sin t + \frac{1}{4}e^{-t} - \frac{3}{20}e^{-t/3}$.

9.6.4. Solve the following initial value problems. In each case, graph the solution and explain what type of motion is represented. (*a*) $\ddot{u} + 25u = 3\cos 4t$, $u(0) = 1$, $\dot{u}(0) = 1$,

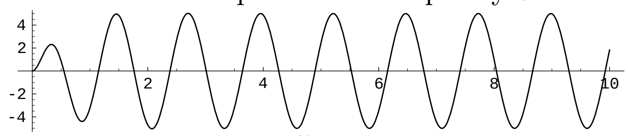
(b) $\ddot{u} + 4\dot{u} + 40u = 125\cos 5t$, $u(0) = 0$, $\dot{u}(0) = 0$, (c) $\ddot{u} + 6\dot{u} + 5u = 25\sin 5t$, $u(0) = 4$, $\dot{u}(0) = 2$, (d) $\ddot{u} + 16u = \sin 4t$, $u(0) = 0$, $\dot{u}(0) = 0$.

Solution:

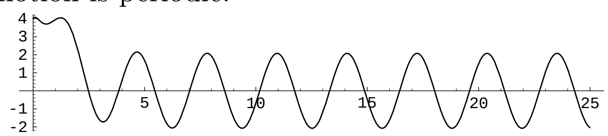
(a) $u(t) = \frac{1}{3}\cos 4t + \frac{2}{3}\cos 5t + \frac{1}{5}\sin 5t$; undamped quasi-periodic motion with fast frequency 4.5 and beat frequency .5:



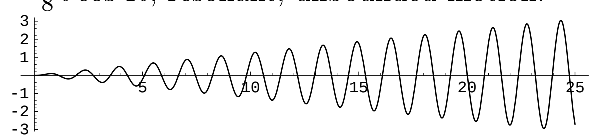
(b) $u(t) = 3\cos 5t + 4\sin 5t - e^{-2t}\left(3\cos 6t + \frac{13}{3}\sin 6t\right)$; the transient is an underdamped motion; the persistent motion is periodic of frequency 5 and amplitude 5:



(c) $u(t) = -\frac{60}{29}\cos 2t + \frac{5}{29}\sin 2t - \frac{56}{29}e^{-5t} + 8e^{-t}$; the transient is an overdamped motion; the persistent motion is periodic:



(d) $u(t) = \frac{1}{32}\sin 4t - \frac{1}{8}t\cos 4t$; resonant, unbounded motion:



9.6.5. A mass $m = 25$ is attached to a unit spring with $k = 1$, and frictional coefficient $\beta = .01$. The spring will break when it moves more than 1 unit. Ignoring the effect of the transient, what is the maximum allowable amplitude α of periodic forcing at frequency $\eta =$ (a) .19 ? (b) .2 ? (c) .21 ?

Solution: In general, by (9.100), the maximal allowable amplitude is $\alpha = \sqrt{m^2(\omega^2 - \eta^2)^2 + \beta^2\eta^2} = \sqrt{625\eta^4 - 49.9999\eta^2 + 1}$, which, in the particular cases is (a) 0.0975, (b) 0.002, (c) 0.1025.

9.6.6. For what range of frequencies η can you force the mass in Exercise 9.6.5 with amplitude $\alpha = .5$ without breaking the spring?

Solution: $\eta \le 0.14142$ or $\eta \ge 0.24495$.

9.6.7. How large should the friction in Exercise 9.6.5 be so that you can safely force the mass with amplitude $\alpha = .5$ at any frequency?

Solution: $\beta \ge 5\sqrt{2 - \sqrt{3}} = 2.58819$.

9.6.8. Suppose the mass-spring-oil system of Exercise 9.5.37(b) is subject to a periodic external force $2\cos 2t$. Discuss, in as much detail as you can, the long term motion of the mass.

Solution: The solution to $.5\ddot{u} + \dot{u} + 2.5u = 2\cos 2t$, $u(0) = 2$, $\dot{u}(0) = 0$, is

$$u(t) = \frac{4}{17}\cos 2t + \frac{16}{17}\sin 2t + e^{-t}\left(\frac{30}{17}\cos 2t - \frac{1}{17}\sin 2t\right)$$
$$= .9701\cos(2t - 1.3258) + 1.7657e^{-t}\cos(2t + .0333).$$

The solution consists of a persistent periodic vibration at the forcing frequency of 2, with a phase lag of $\tan^{-1} 4 = 1.32582$ and amplitude $4/\sqrt{17} = .97014$, combined with a transient vibration at the same frequency with exponentially decreasing amplitude.

♢ 9.6.9. Show that the conclusions based on (9.96) do not depend upon the choice of initial conditions (9.95).

Solution: ■

♢ 9.6.10. Write down the solution $u(t, \eta)$ to the initial value problem

$$m \frac{d^2 u}{dt^2} + k\, u = \alpha \cos \eta\, t, \qquad u(0) = \dot u(0) = 0,$$

for (a) a non-resonant forcing function at frequency $\eta \neq \omega$; (b) a resonant forcing function at frequency $\eta = \omega$. (c) Show that, as $\eta \to \omega$, the limit of the non-resonant solution equals the resonant solution. Conclude that the solution $u(t, \eta)$ depends continuously on the frequency η even though its mathematical formula changes significantly at resonance.

Solution: (a) $u(t) = \dfrac{\alpha(\cos \eta\, t - \cos \omega\, t)}{m(\omega^2 - \eta^2)}$; (b) $u(t) = \dfrac{\alpha\, t}{2 m \omega} \sin \omega\, t$;
(c) Use l'Hôpital's rule, differentiating with respect to η to compute

$$\lim_{\eta \to \omega} \frac{\alpha(\cos \eta\, t - \cos \omega\, t)}{m(\omega^2 - \eta^2)} = \lim_{\eta \to \omega} \frac{\alpha\, t \sin \eta\, t}{2 m \eta} = \frac{\alpha\, t}{2 m \omega} \sin \omega\, t.$$

♢ 9.6.11. Justify the solution formulae (9.100) and (9.101).

Solution: Using the method of undetermined coefficients, we set

$$u^\star(t) = A \cos \eta\, t + B \sin \eta\, t.$$

Substituting into the differential equation (9.99), and then equating coefficients of $\cos \eta\, t, \sin \eta\, t$, we find

$$m\,(\omega^2 - \eta^2)\, A + \beta \eta\, B = \alpha, \qquad - \beta \eta\, A + m\,(\omega^2 - \eta^2)\, B = 0,$$

where we replaced $k = m \omega^2$. Thus,

$$A = \frac{\alpha\, m\,(\omega^2 - \eta^2)}{m^2 (\omega^2 - \eta^2)^2 + \beta^2 \eta^2}, \qquad B = \frac{\alpha \beta \eta}{m^2 (\omega^2 - \eta^2)^2 + \beta^2 \eta^2}\,.$$

We then put the resulting solution in phase-amplitude form

$$u^\star(t) = a \cos(\eta\, t - \varepsilon),$$

where, according to (2.7), $A = a \cos \varepsilon, B = a \sin \varepsilon$, which implies (9.100–101).

9.6.12. Classify the following RLC circuits as (i) underdamped, (ii) critically damped, or (iii) overdamped: (a) $R = 1,\ L = 2,\ C = 4$, (b) $R = 4,\ L = 3,\ C = 1$, (c) $R = 2,\ L = 3,\ C = 3$, (d) $R = 4,\ L = 10,\ C = 2$, (e) $R = 1,\ L = 1,\ C = 3$.

Solution:
(a) underdamped, (b) overdamped, (c) critically damped, (d) underdamped, (e) underdamped.

9.6.13. Find the current in each of the unforced RLC circuits in Exercise 9.6.12 induced by the initial data $u(0) = 1,\ \dot u(0) = 0$.

Solution:
- (a) $u(t) = e^{-t/4} \cos \frac{1}{4} t + e^{-t/4} \sin \frac{1}{4} t$,
- (b) $u(t) = \frac{3}{2} e^{-t/3} - \frac{1}{2} e^{-t}$,
- (c) $u(t) = e^{-t/3} + \frac{1}{3} t e^{-t/3}$,
- (d) $u(t) = e^{-t/5} \cos \frac{1}{10} t + 2 e^{-t/5} \sin \frac{1}{10} t$,

(e) $u(t) = e^{-t/2} \cos \frac{1}{2\sqrt{3}}\, t + \sqrt{3}\, e^{-t/2} \sin \frac{1}{2\sqrt{3}}\, t.$

9.6.14. A circuit with $R = 1, L = 2, C = 4$ includes an alternating current source $F(t) = 25 \cos 2t$. Find the solution to the initial value problem $u(0) = 1, \dot{u}(0) = 0$.

Solution:

$$\begin{aligned} u(t) &= \tfrac{165}{41}\, e^{-t/4} \cos \tfrac{1}{4}\, t - \tfrac{91}{41}\, e^{-t/4} \sin \tfrac{1}{4}\, t - \tfrac{124}{41} \cos 2t + \tfrac{32}{41} \sin 2t \\ &= 4.0244\, e^{-.25t} \cos .25t - 2.2195\, e^{-.25t} \sin .25t - 3.0244 \cos 2t + .7805 \sin 2t. \end{aligned}$$

9.6.15. A superconducting LC circuit has no resistance: $R = 0$. Discuss what happens when the circuit is wired to an alternating current source $F(t) = \alpha \cos \eta\, t$.

Solution: The natural vibrational frequency is $\omega = 1/\sqrt{RC}$. If $\eta \neq \omega$ then the circuit experiences a quasi-periodic vibration as a combination of the two frequencies. As η gets close to ω, the current amplitude becomes larger an larger, exhibiting beats. When $\eta = \omega$, the circuit is in resonance, and the current amplitude grows without bound.

9.6.16. A circuit with $R = .002$, $L = 12.5$, and $C = 50$ can carry a maximum current of 250. Ignoring the effect of the transient, what is the maximum allowable amplitude α of an applied periodic current $F(t) = \alpha \cos \eta\, t$ at frequency $\eta =$ (a) .04 ? (*b*) .05 ? (*c*) .1 ?

Solution: (a) .02, (*b*) 2.8126, (*c*) 26.25.

9.6.17. Given the circuit in Exercise 9.6.16, what range of frequencies η can you supply a unit amplitude periodic current source?

Solution: $\eta \leq .03577$ or $\eta \geq .04382$.

9.6.18. How large should the resistance in the circuit in Exercise 9.6.16 be so that you can safely apply any unit amplitude periodic current?

Solution: $R \geq .10051$.

9.6.19. Find the general solution to the following forced second order systems:

(a) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 7 & -2 \\ -2 & 4 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 0 \end{pmatrix}$, (*b*) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 5 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ 5 \sin 3t \end{pmatrix}$,

(*c*) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 13 & -6 \\ -6 & 8 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 5 \cos 2t \\ \cos 2t \end{pmatrix}$, (*d*) $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos \frac{1}{2} t \\ -\cos \frac{1}{2} t \end{pmatrix}$,

(*e*) $\begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 4 & -2 \\ -2 & 3 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 11 \sin 2t \end{pmatrix}$, (*f*) $\dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix} \mathbf{u} = \begin{pmatrix} \cos t \\ 0 \\ \cos t \end{pmatrix}$,

(g) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix} \dfrac{d^2\mathbf{u}}{dt^2} + \begin{pmatrix} 5 & -1 & -1 \\ -1 & 6 & 3 \\ -1 & 3 & 9 \end{pmatrix} \mathbf{u} = \begin{pmatrix} 0 \\ \cos t \\ \cos t \end{pmatrix}$.

Solution:

(a) $\mathbf{u}(t) = \cos t \begin{pmatrix} \frac{3}{14} \\ \frac{1}{7} \end{pmatrix} + r_1 \cos(2\sqrt{2}\, t - \delta_1) \begin{pmatrix} -2 \\ 1 \end{pmatrix} + r_2 \cos(\sqrt{3}\, t - \delta_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(*b*) $\mathbf{u}(t) = \sin 3t \begin{pmatrix} \frac{1}{2} \\ -1 \end{pmatrix} + r_1 \cos(\sqrt{4 + \sqrt{5}}\, t - \delta_1) \begin{pmatrix} -1 - \sqrt{5} \\ 2 \end{pmatrix} + r_2 \cos(4 - \sqrt{5}\, t - \delta_2) \begin{pmatrix} -1 + \sqrt{5} \\ 2 \end{pmatrix}$.

(*c*) $\mathbf{u}(t) = \begin{pmatrix} \frac{1}{2} t \sin 2t + \frac{1}{3} \cos 2t \\ \frac{3}{4} t \sin 2t \end{pmatrix} + r_1 \cos\left(\sqrt{17}\, t - \delta_1 \right) \begin{pmatrix} -3 \\ 2 \end{pmatrix} + r_2 \cos(2t - \delta_2) \begin{pmatrix} 2 \\ 3 \end{pmatrix}$,

$$(d)\ \mathbf{u}(t) = \cos \tfrac{1}{2}t \begin{pmatrix} \frac{2}{17} \\ -\frac{12}{17} \end{pmatrix} + r_1 \cos(\tfrac{\sqrt{5}}{\sqrt{3}}\, t - \delta_1) \begin{pmatrix} -3 \\ 1 \end{pmatrix} + r_2 \cos(\tfrac{1}{\sqrt{2}}\, t - \delta_2) \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

$$(e)\ \mathbf{u}(t) = \cos t \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix} + \sin 2t \begin{pmatrix} \frac{1}{6} \\ -\frac{2}{3} \end{pmatrix} + r_1 \cos(\tfrac{\sqrt{8}}{\sqrt{5}}\, t - \delta_1) \begin{pmatrix} -5 \\ 2 \end{pmatrix} + r_2 \cos(\tfrac{1}{\sqrt{3}}\, t - \delta_2) \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$(f)\ \mathbf{u}(t) = \cos t \begin{pmatrix} \frac{6}{11} \\ \frac{5}{11} \\ \frac{1}{11} \end{pmatrix} + r_1 \cos(\sqrt{12}\, t - \delta_1) \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + r_2 \cos(3t - \delta_2) \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + r_3 \cos(\sqrt{2}\, t - \delta_3) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

$$(g)\ \mathbf{u}(t) = \cos t \begin{pmatrix} \frac{1}{8} \\ \frac{3}{8} \\ 0 \end{pmatrix} + r_1 \cos(\sqrt{3}\, t - \delta_1) \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} + r_2 \cos(\sqrt{2}\, t - \delta_2) \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + r_3 \cos(t - \delta_3) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

9.6.20. (*a*) Find the resonant frequencies of a mass-spring chain consisting of two masses, $m_1 = 1$ and $m_2 = 2$ connected to top and bottom supports by identical springs with unit stiffness. (*b*) Write down an explicit forcing function that will excite the resonance.

Solution:

(*a*) The resonant frequencies are $\sqrt{\frac{3-\sqrt{3}}{2}} = .796225$, $\sqrt{\frac{3+\sqrt{3}}{2}} = 1.53819$.

(*b*) For example, a forcing function of the form $\cos\left(\sqrt{\frac{3+\sqrt{3}}{2}}\, t\right) \mathbf{w}$ where $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is not orthogonal to the eigenvector $\begin{pmatrix} -1-\sqrt{3} \\ 1 \end{pmatrix}$, so $w_2 \neq (1+\sqrt{3}\,)w_1$, will excite resonance.

9.6.21. Suppose one of the supports is removed from the mass-spring chain of Exercise 9.6.20. Does your forcing function still excite the resonance? Do the internal vibrations of the masses (*i*) speed up, (*ii*) slow down, (*iii*) or remain the same? Does your answer depend upon which of the two supports is removed?

Solution: When the bottom support is removed, the resonant frequencies are $\frac{\sqrt{5-\sqrt{17}}}{2} = .468213$, $\frac{\sqrt{5+\sqrt{17}}}{2} = 1.51022$. When the top support is removed, the resonant frequencies are $\sqrt{\frac{2-\sqrt{2}}{2}} = .541196$, $\sqrt{\frac{2+\sqrt{2}}{2}} = 1.30656$. In both cases the vibrations are slower. The previous forcing function will not excite resonance.

♣ 9.6.22. Find the resonant frequencies of the following structures, assuming the nodes all have unit mass. Then find a means of forcing the structure at one of the resonant frequencies, and yet not exciting the resonance. Can you also force the structure without exciting any mechanism or rigid motion? (*a*) the square truss of Exercise 6.3.5; (*b*) the joined square truss of Exercise 6.3.6; (*c*) the house of Exercise 6.3.9; (*d*) the space station in Exercise 6.3.12; (*e*) the triatomic molecule of Example 9.37; (*f*) the water molecule of Exercise 9.5.29.

Solution: In each case, you need to force the system by $\cos(\omega\, t)\mathbf{a}$ where $\omega^2 = \lambda$ is an eigenvalue and $\mathbf{a}$ is orthogonal to the corresponding eigenvector. In order not to excite an instability, $\mathbf{a}$ needs to also be orthogonal to the kernel of the stiffness matrix spanned by the unstable mode vectors.

(*a*) Resonant frequencies: $\omega_1 = .5412, \omega_2 = 1.1371, \omega_3 = 1.3066, \omega_4 = 1.6453$. Eigenvectors:

$\mathbf{v}_1 = \begin{pmatrix} -.6533 \\ -.2706 \\ -.6533 \\ .2706 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -.2706 \\ -.6533 \\ .2706 \\ -.6533 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -.2706 \\ .6533 \\ -.2706 \\ -.6533 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -.6533 \\ .2706 \\ .6533 \\ .2706 \end{pmatrix}$. No unstable modes.

(*b*) Resonant frequencies: $\omega_1 = .4209, \omega_2 = 1$ (double), $\omega_3 = 1.2783, \omega_4 = 1.6801, \omega_5 = 1.8347$. Eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} -.6626 \\ -.1426 \\ -.6626 \\ .1426 \\ -.2852 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ -.6862 \\ 0 \\ -.4425 \\ .1218 \\ -.5643 \end{pmatrix}, \widehat{\mathbf{v}}_2 = \begin{pmatrix} 0 \\ -.4425 \\ 0 \\ .6862 \\ .5643 \\ .1218 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -.5 \\ -.2887 \\ .5 \\ -.2887 \\ 0 \\ .5774 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} .2470 \\ -.3825 \\ .2470 \\ .3825 \\ -.7651 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} .5 \\ -.2887 \\ -.5 \\ -.2887 \\ 0 \\ .5774 \end{pmatrix}$. No unstable modes.

(*c*) Resonant frequencies: $\omega_1 = .3542, \omega_2 = .9727, \omega_3 = 1.0279, \omega_4 = 1.6894, \omega_5 = 1.7372$. Eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} -.0989 \\ -.0706 \\ 0 \\ -.9851 \\ .0989 \\ -.0706 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -.1160 \\ .6780 \\ .2319 \\ 0 \\ -.1160 \\ -.6780 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} .1251 \\ -.6940 \\ 0 \\ .0744 \\ -.1251 \\ -.6940 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} .3914 \\ .2009 \\ -.7829 \\ 0 \\ .3914 \\ -.2009 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} .6889 \\ .1158 \\ 0 \\ -.1549 \\ -.6889 \\ .1158 \end{pmatrix}$. Unstable mode: $\mathbf{z} = (1, 0, 1, 0, 1, 0)^T$.

(*d*) Resonant frequencies: $\omega_1 = 1, \omega_2 = \sqrt{3} = 1.7372$. Eigenvectors: $\mathbf{v}_1 = (1, 0, -1)^T, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Rigid motion: $\mathbf{z} = (1, 1, 1)^T$.

(*e*) ■ Resonant frequencies: $\omega_1 = 1, \omega_2 = \sqrt{3} = 1.7372$. Eigenvectors: $\mathbf{v}_1 = (1, 0, -1)^T, \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. Unstable mode: $\mathbf{z} = (1, 1, 1)^T$.
