HW solution

1.8.1 (a)

$$x - 2y = 1$$
$$3x + 2y = -3$$

(i) $A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$ $\quad \vec{b} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 3 & 2 & -3 \end{array}\right) \qquad L_{21} = 3$$

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 8 & -6 \end{array}\right) \qquad U = \begin{pmatrix} 1 & -2 \\ 0 & 8 \end{pmatrix}$$

Unique Solution. $\qquad L = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 3 & 2 & b_2 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 0 & 8 & b_2 - 3b_1 \end{array}\right)$$

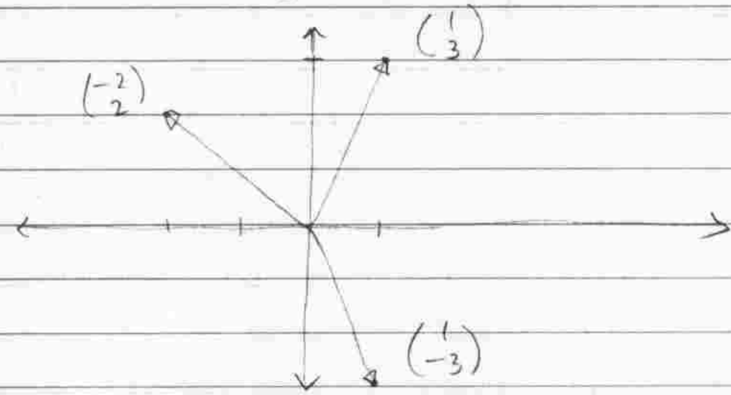
Solution exists! for any b_1 & b_2.

(iii)

There are no null vectors, as the solution is unique. Therefore there is no basis of kernel.

(iv) Aiming = $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$

Target = $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(V) There are kernel basis vectors.

(b)
$$2x + y + 3z = 1$$
$$x + 4y - 2z = -3$$

(i) $\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 1 & 4 & -2 & -3 \end{array}\right)$ $L_{21} = \frac{1}{2}$

$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 7/2 & -7/2 & -7/2 \end{array}\right)$

$U = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 7/2 & -7/2 \end{pmatrix}$ L = does not exist.

There is one free variable.
There are infinitely many solutions.

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 1 & 4 & -2 & b_2 \end{array}\right) \qquad L_{21} = \frac{1}{2}$$

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & b_1 \\ 0 & 7/2 & -7/2 & b_2 - \frac{b_1}{2} \end{array}\right)$$

For any b_1, b_2 infinite solutions exist.

(iii) Let z be free.

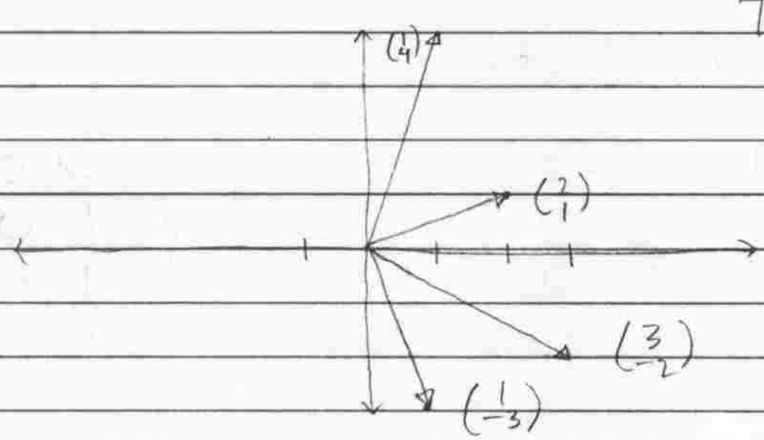
$$7y - 7z = -7 \qquad 2x + y + 3z = 1$$
$$y - z = -1 \qquad 2x + 4z - 1 = 1$$
$$y = z - 1 \qquad 2x = -4z + 2$$
$$x = -2z + 1$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1-2z \\ -1+z \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

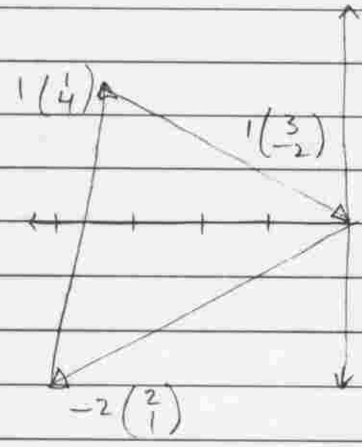
$$-2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ 4 \end{pmatrix} + 1\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$ kernel basis of A

(iv) Aiming: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

Target: $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$



(v)



(c)

$$x + y - 2z = -3$$
$$2x - y + 3z = 7$$
$$x - 2y + 5z = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 2 & -1 & 3 & 7 \\ 1 & -2 & 5 & 1 \end{array}\right) \qquad L_{21} = 2,\quad L_{31} = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & -3 & 7 & 13 \\ 0 & -3 & 7 & 4 \end{array}\right) \qquad L_{32} = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & -3 & 7 & 13 \\ 0 & 0 & 0 & -9 \end{array}\right)$$

No Solution

$$U = \begin{pmatrix} 1 & 1 & -2 \\ 0 & -3 & 7 \\ 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & b_1 \\ 2 & -1 & 3 & b_2 \\ 1 & -2 & 5 & b_3 \end{array}\right) \quad L_{21} = 2, \quad L_{31} = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & b_1 \\ 0 & -3 & 7 & b_2 - 2b_1 \\ 0 & -3 & 7 & b_3 - b_1 \end{array}\right) \quad L_{32} = 1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & b_1 \\ 0 & -3 & 7 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 - b_2 + b_1 \end{array}\right)$$

$b_3 - b_2 + b_1 = 0$ then infinite solutions.
$b_3 - b_2 + b_1 \neq 0$ then no solution.

(iii) Let z be free

$$-3y + 7z = 13 \qquad\qquad x + y - 2z = -3$$

$$y = \frac{13 - 7z}{-3} \qquad\qquad x + \frac{13 - 7z}{-3} - 2z = -3$$

$$-3x + 13 - 7z + 6z = 9$$

$$-3x = z - 4$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{4}{3} - \frac{z}{3} \\ -\frac{13}{3} + \frac{7}{3}z \\ z \end{pmatrix} \qquad x = \frac{z - 4}{-3}$$

$$= \begin{pmatrix} 4/3 \\ -13/3 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1/3 \\ 7/3 \\ 1 \end{pmatrix} \longrightarrow \text{basis kernel}$$

$$-\frac{1}{3}\begin{pmatrix}1\\2\\1\end{pmatrix} + \frac{7}{3}\begin{pmatrix}1\\-1\\-2\end{pmatrix} + 1\begin{pmatrix}-2\\3\\5\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

(iv) Aiming :- $\begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\-2\end{pmatrix}, \begin{pmatrix}-2\\3\\5\end{pmatrix}$

Target :- $\begin{pmatrix}-3\\7\\1\end{pmatrix}$

Difficult to draw in 3D.

(v) " " " " "

1.8.2 (a)

$$6x_1 + 3x_2 = 12$$
$$4x_1 + 2x_2 = 9$$

(i) $\left(\begin{array}{cc|c}6 & 3 & 12\\4 & 2 & 9\end{array}\right)$ $L_{21} = \frac{2}{3}$

$$\left(\begin{array}{cc|c}6 & 3 & 12\\0 & 0 & 1\end{array}\right)$$

No Solution.

$U = \begin{pmatrix}6 & 3\\0 & 0\end{pmatrix}$ $L = \begin{pmatrix}1 & 0\\2/3 & 1\end{pmatrix}$

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 6 & 3 & b_1 \\ 4 & 2 & b_2 \end{array}\right) \qquad L_{21} = \frac{2}{3}$$

$$\left(\begin{array}{cc|c} 6 & 3 & b_1 \\ 0 & 0 & b_2 - \frac{2}{3} b_1 \end{array}\right)$$

if $\quad b_2 - \frac{2}{3} b_1 = 0 \qquad$ Infinite Solutions

$\quad\;\; b_2 - \frac{2}{3} b_1 \neq 0 \qquad$ No Solution.

(iii) Let y be free.

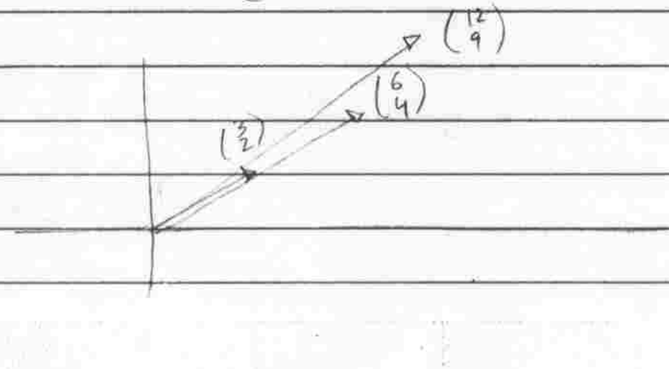
$$6x + 3y = 12 \;)\div 3$$
$$2x + y = 4$$
$$x = \frac{4-y}{2}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 - \frac{y}{2} \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$$

Basis Kernel

$$-\frac{1}{2}\begin{pmatrix} 6 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(iv) Aiming -: $\begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ Target -: $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$



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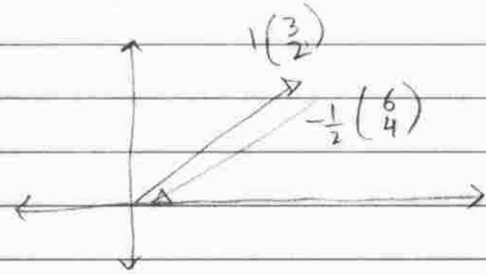


(b)

$$8x_1 + 12x_2 = 16$$
$$6x_1 + 9x_2 = 13$$

(i) $\left(\begin{array}{cc|c} 8 & 12 & 16 \\ 6 & 9 & 13 \end{array}\right)$ $\quad L_{21} = \frac{3}{4}$

$$\left(\begin{array}{cc|c} 8 & 12 & 16 \\ 0 & 0 & 1 \end{array}\right)$$

No Solution

$$U = \begin{pmatrix} 8 & 12 \\ 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & 0 \\ 3/4 & 1 \end{pmatrix}$$

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left(\begin{array}{cc|c} 8 & 12 & b_1 \\ 6 & 9 & b_2 \end{array}\right) \qquad L_{21} = \frac{3}{4}$$

$$\left(\begin{array}{cc|c} 8 & 12 & b_1 \\ 0 & 0 & b_2 - \frac{3}{4}b_1 \end{array}\right)$$

if $\quad b_2 - \frac{3}{4}b_1 = 0 \quad$ Infinite Solutions.

if $\quad b_2 - \frac{3}{4}b_1 \neq 0 \quad$ No Solution

(iii) Let y be free.

$$8x + 12y = 16 \;]\div 4$$

$$2x + 3y = 4$$

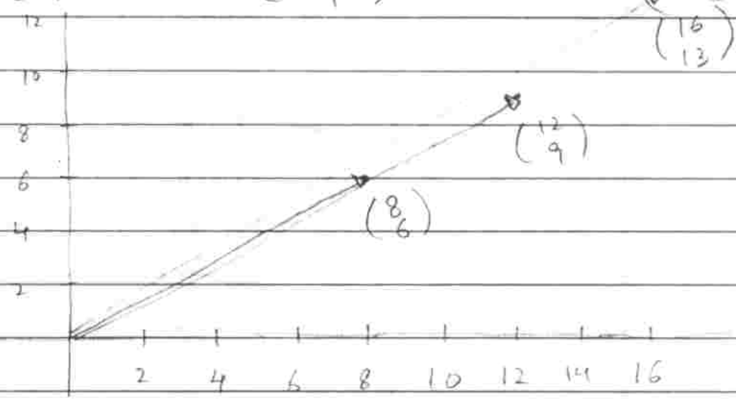
$$x = \frac{4-3y}{2} = 2 - \frac{3}{2}y$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 - \frac{3}{2}y \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + y\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

Basis Kernel

$$-\frac{3}{2}\begin{pmatrix} 8 \\ 6 \end{pmatrix} + 1\begin{pmatrix} 12 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(iv) Aiming: $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ & $\begin{pmatrix} 12 \\ 9 \end{pmatrix}$ Target: $\begin{pmatrix} 16 \\ 13 \end{pmatrix}$



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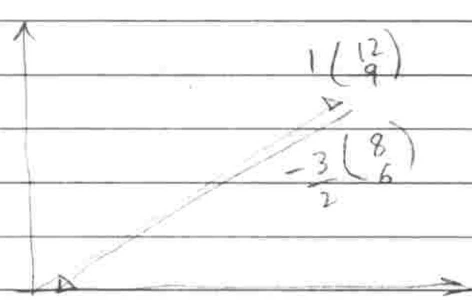


(c) $x_1 + 2x_2 = 1$
$2x_1 + 5x_2 = 2$
$3x_1 + 6x_2 = 3$

(i) $\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 3 & 6 & 3 \end{array}\right)$ $L_{21} = 2$, $L_{31} = 3$

$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

Infinite Solutions.

$U = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ L = Does not exist.

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 2 & 5 & b_2 \\ 3 & 6 & b_3 \end{array}\right)$ $L_{21} = 2$, $L_{31} = 3$

$\left(\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 - 3b_1 \end{array}\right)$

if $b_3 - 3b_1 = 0$ Infinite Solutions.
$b_3 - 3b_1 \neq 0$ No Solutions.

(iii) No free variable $\Rightarrow$ No Null vectors
So, no kernel basis.

(iv) Aiming $= \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ & $\begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$ Target: $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Difficult to draw in 3D.

(v) " " " "

(d) $$2x_1 - 6x_2 + 4x_3 = 2$$
$$-x_1 + 3x_2 - 2x_3 = -1$$

(i) $\left(\begin{array}{ccc|c} 2 & -6 & 4 & 2 \\ -1 & 3 & -2 & -1 \end{array}\right)$

rewriting : exchanging rows

$\left(\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ 2 & -6 & 4 & 2 \end{array}\right)$ $L_{21} = -2$

$\left(\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Infinite Solutions.

$U = \begin{pmatrix} -1 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix}$ L = Does not exist.

(ii) $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 3 & -2 & b_2 \\ 2 & -6 & 4 & b_1 \end{array}\right) \quad L_{21} = -2$$

$$\left(\begin{array}{ccc|c} -1 & 3 & -2 & b_2 \\ 0 & 0 & 0 & b_1 + 2b_2 \end{array}\right)$$

if $b_1 + 2b_2 = 0$ Infinite Solution
$b_1 + 2b_2 \neq 0$ No Solution

(iii) Let y & z be free.

$$-x + 3y - 2z = -1$$

$$x = 3y - 2z + 1$$

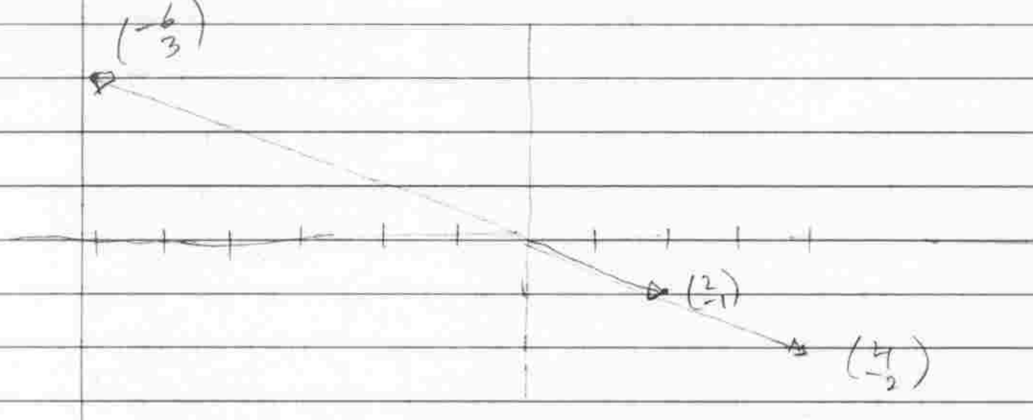
$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3y - 2z + 1 \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Kernel Basis

$$3 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 1 \begin{pmatrix} 3 \\ -6 \end{pmatrix} + 0 \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} 3 \\ -6 \end{pmatrix} + 1 \begin{pmatrix} -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(iv) Aiming: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ Target: $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$



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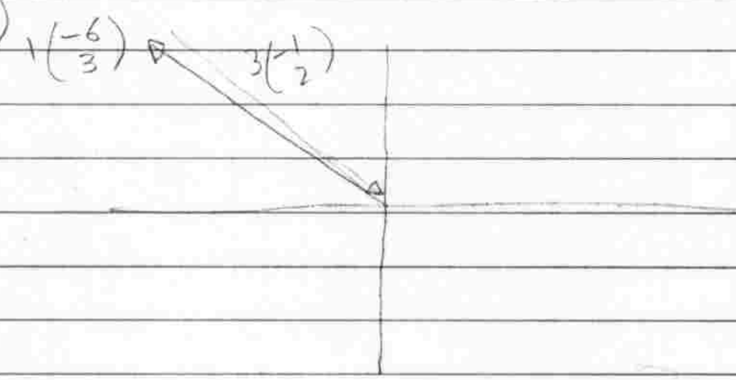

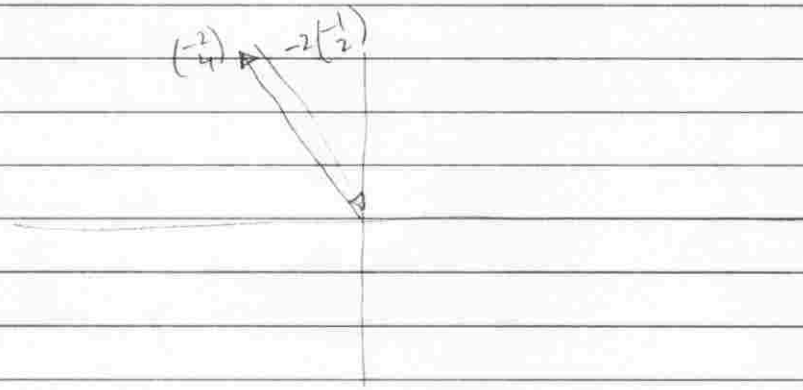

(f)

$$x_1 + x_2 + x_3 + 9x_4 = 8$$
$$x_2 + 2x_3 + 8x_4 = 7$$
$$-3x_1 + x_3 - 7x_4 = 9$$

(i)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ -3 & 0 & 1 & -7 & 9 \end{array}\right) \quad \begin{array}{l} L_{21} = 0 \\ L_{31} = -3 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ 0 & 3 & 4 & 20 & 33 \end{array}\right) \quad L_{32} = 3$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 9 & 8 \\ 0 & 1 & 2 & 8 & 7 \\ 0 & 0 & -2 & -4 & 12 \end{array}\right)$$

$$U = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & -2 & -4 \end{pmatrix} \quad L = \text{Does not exist}$$

Infinite solutions.

(ii)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 9 & b_1 \\ 0 & 1 & 2 & 8 & b_2 \\ 0 & 0 & -2 & -4 & b_3 + 3b_1 - 3b_2 \end{array}\right)$$

for any values, there are always infinite solutions.

(iii) Let x_4 be free.

$$2x_3 + 4x_4 = -12 \;\Big]\div 2 \qquad x_3 + 2x_4 = -6$$
$$x_3 = -6 - 2x_4$$

$$x_2 + 2x_3 + 8x_4 = 7$$
$$x_2 = 7 - 8x_4 + 12 + 4x_4 = 19 - 4x_4$$

$$x_1 + x_2 + x_3 + 9x_4 = 7$$

$$x_1 = 7 - 9x_4 - 19 + 4x_4 + 6 + 2x_4$$
$$x_1 = -6 - 3x_4$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6-3x_4 \\ 19-4x_4 \\ -6-2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6 \\ 19 \\ -6 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ -4 \\ -2 \\ 1 \end{pmatrix}$$

$$-3\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + (-4)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (-2)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 9 \\ 8 \\ -7 \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(iv) Aiming: $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ -7 \end{pmatrix}$ Target: $\begin{pmatrix} 8 \\ 7 \\ 9 \end{pmatrix}$

Difficult to draw in 3D

(v) " " " " "