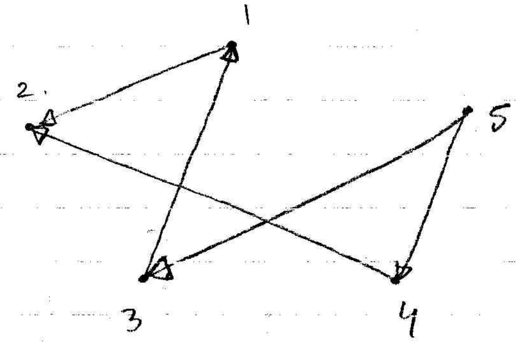
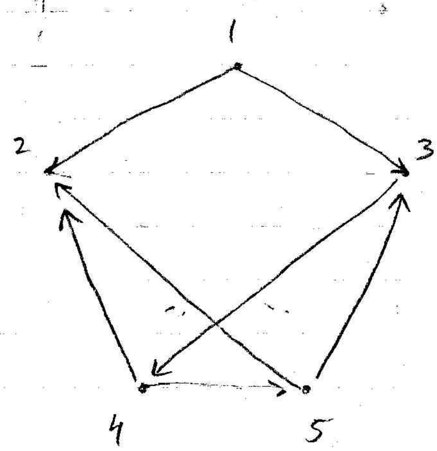
Math 410
Solutions

2.6.1 (d)

$$\begin{pmatrix} -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$



2.6.4 (d)



$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

basis of cokernel of A = null vectors of A^T

$$A^T = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \qquad L_{21} = -1$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \qquad L_{32} = -1$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \qquad L_{43} = -1$$

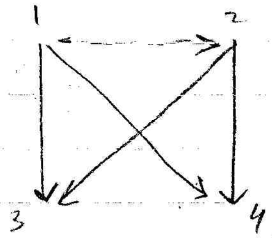
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 \end{pmatrix} \qquad L_{54} = -1$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

free variables

No. of independent circuits = 2

2.6.5 (a)



$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

(b) Rank $(A) = 3$

Row reduced $= \begin{pmatrix} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

(c) $\dim(\ker) = 1$ $\quad$ $\dim(\text{coker}) = 2$

$\dim(\text{range}) = 3$ $\quad$ $\dim(\text{corange}) = 3$

(d) Let d be free.

$c - d = 0$ $\quad$ $b - c = 0$ $\quad$ $a - c = 0$

$c = d$ $\quad$ $b = c = d$ $\quad$ $a = c = d$

$$\vec{a} = \begin{pmatrix} d \\ d \\ d \\ d \end{pmatrix} = d \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

↳ Basis of kernel.

$A^T =$ reduced $\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Let d & e be free.

$c + e = 0$ $\quad$ $b - d - e = 0$

$c = -e$ $\quad$ $b = d + e$

$a + b + c = 0$

$a = -d - e + e = -d$

$$\vec{a} = \begin{pmatrix} -d \\ d+e \\ -e \\ d \\ e \end{pmatrix} = d \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + e \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

basis of cokernel.

(e) $Ax = b$

$$A = \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & b_1 \\ 1 & -1 & 0 & 0 & b_2 \\ 1 & 0 & 0 & -1 & b_3 \\ 0 & 1 & -1 & 0 & b_4 \\ 0 & 1 & 0 & -1 & b_5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & b_1 \\ 0 & -1 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & -1 & b_3 - b_1 \\ 0 & 1 & -1 & 0 & b_4 \\ 0 & 1 & 0 & -1 & b_5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & b_1 \\ 0 & -1 & 1 & 0 & b_2 - b_1 \\ 0 & 0 & 1 & -1 & b_3 - b_1 \\ 0 & 0 & 0 & 0 & b_4 + b_2 - b_1 \\ 0 & 0 & 1 & -1 & b_5 + b_2 - b_1 \end{array}\right)$$

$$\left(\begin{array}{cccc|l} 1 & 0 & -1 & 0 & b_1 \\ 0 & -1 & 1 & 0 & b_2-b_1 \\ 0 & 0 & 1 & -1 & b_3-b_1 \\ 0 & 0 & 0 & 0 & b_4+b_2-b_1 \\ 0 & 0 & 0 & 0 & b_5+b_2-b_1-b_3+b_1 \end{array}\right)$$

$b_4+b_2-b_1 \neq 0$ & $b_5+b_2-b_3 \neq 0$

then, no solution.

(f) $\vec{b} = \begin{pmatrix} 10 \\ 5 \\ 10 \\ 5 \\ 5 \end{pmatrix}$ gives a solution for $Ax=b$

$\vec{b} = \begin{pmatrix} b_1 \\ b_1-b_4 \\ b_3 \\ b_4 \\ b_3-b_2 \end{pmatrix}$

6.1.4 (a) $c_1 = 1 \quad c_2 = \frac{1}{2} \quad c_3 = \frac{2}{3} \quad c_4 = \frac{1}{2} \quad c_5 = 1$

$f = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad K = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{7}{6} & -\frac{2}{3} & 0 \\ 0 & -\frac{2}{3} & \frac{7}{6} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$

$Ku = f$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{7}{6} & -\frac{2}{3} & 0 & 1 \\ 0 & -\frac{2}{3} & \frac{7}{6} & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right) \qquad L_{21} = -\frac{1}{3}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & -\frac{2}{3} & \frac{7}{6} & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right) \qquad L_{32} = -\frac{2}{3}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & \frac{13}{18} & -\frac{1}{2} & \frac{5}{3} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \end{array}\right) \quad L_{43} = \frac{-9}{13}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & \frac{13}{18} & -\frac{1}{2} & \frac{5}{3} \\ 0 & 0 & 0 & \frac{15}{13} & \frac{45}{39} \end{array}\right)$$

$$\frac{15}{13} u_4 = \frac{45}{39} \qquad \frac{13}{18} u_3 - \frac{1}{2} u_4 = \frac{5}{3}$$

$$u_4 = \frac{3}{3} = 1 \qquad \frac{13}{18} u_3 = \frac{5}{3} + \frac{1}{2} \Rightarrow u_3 = \frac{18}{13} \times \frac{13}{6} = 3$$

$$u_2 - \frac{2}{3} u_3 = 1 \Rightarrow u_2 = \frac{2}{3} \times 3 + 1 = 3$$

$$\frac{3}{2} u_1 - \frac{1}{2} u_2 = 0 \qquad \frac{3}{2} u_1 = \frac{1}{2} \times 3 \Rightarrow u_1 = 1$$

$$u = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

$$y = Au = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -2 \\ -1 \end{pmatrix}$$

(b)

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{7}{6} & -\frac{2}{3} & 0 & 1 \\ 0 & -\frac{2}{3} & \frac{7}{6} & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \qquad L_{21} = -\frac{1}{3}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & -\frac{2}{3} & \frac{7}{6} & -\frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \qquad L_{32} = -\frac{2}{3}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & \frac{13}{18} & -\frac{1}{2} & \frac{5}{3} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{array}\right) \qquad L_{43} = -\frac{9}{13}$$

$$\left(\begin{array}{cccc|c} \frac{3}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 1 \\ 0 & 0 & \frac{13}{18} & -\frac{1}{2} & \frac{5}{3} \\ 0 & 0 & 0 & \frac{2}{13} & \frac{45}{39} \end{array}\right)$$

$\frac{2}{13} u_4 = \frac{45}{39} \Rightarrow u_4 = \frac{15}{2}$

$\frac{13}{18} u_3 - \frac{1}{2} u_4 = \frac{5}{3}$

$\frac{13}{18} u_3 = \frac{5}{3} + \frac{15}{4}$

$u_3 = \frac{18}{13} \times \frac{65}{12}$

$u_3 = \frac{195}{26}$

$u_2 - \frac{2}{3} u_3 = 1$

$u_2 = 1 + \frac{2}{3} \times \frac{195}{26}$

$u_2 = 1 + \frac{65}{13} = \frac{78}{13}$

$\frac{3}{2} u_1 - \frac{1}{2} u_2 = 0$

$\frac{3}{2} u_1 = \frac{1}{2} \times \frac{78}{13}$

$u_1 = \frac{2}{3} \times \frac{1}{2} \times \frac{78}{13} = \frac{26}{13} = 2$

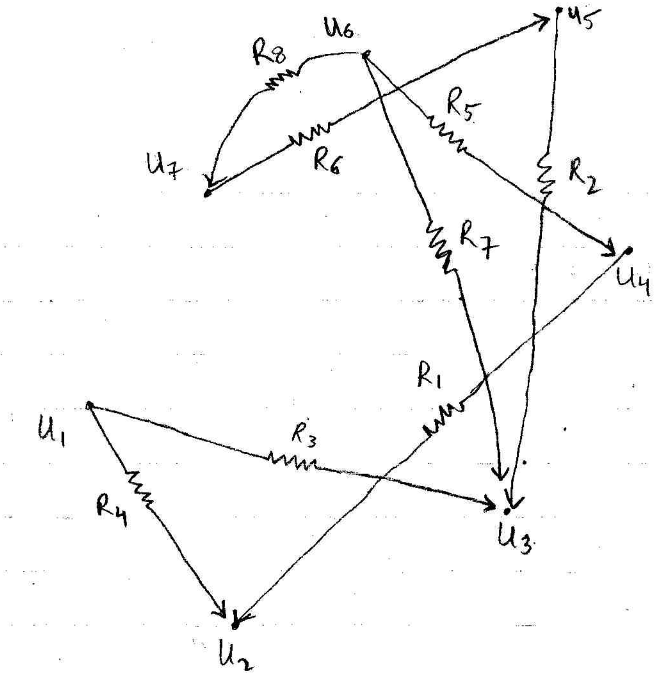
$$u = \begin{pmatrix} 2 \\ 78/13 \\ 195/26 \\ 15/2 \end{pmatrix}$$

$$y = Au =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 78/13 \\ 195/26 \\ 15/2 \end{pmatrix}$$

$$y = \begin{pmatrix} 2 \\ 52/13 \\ 3/2 \\ 0 \\ -15/2 \end{pmatrix}$$

6.2.1 (e)



6.2.4 (a)

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$c_1 = 1 \quad c_2 = 1 \quad c_3 = 1 \quad c_4 = 1/2 \quad c_5 = 1 \quad c_6 = 1 \quad c_7 = 1$

$K = A^T C A$

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} A$$