STRUCTURES PROBLEMS (MATH 410)

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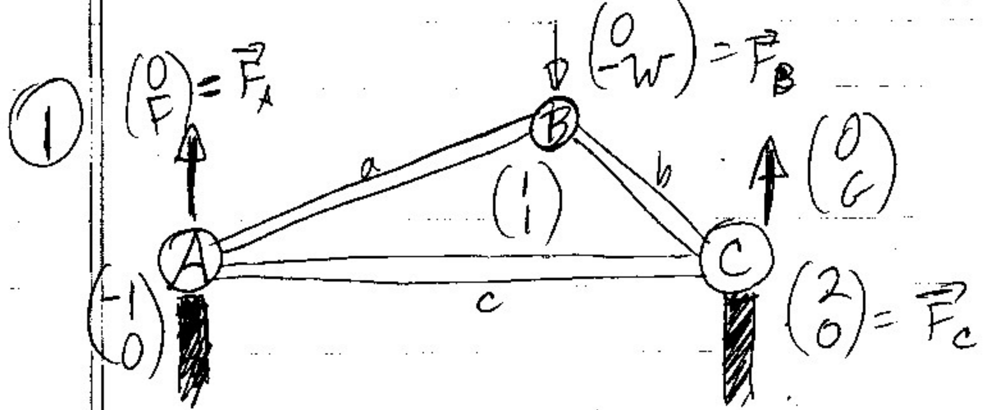

$W = 600$

The equations for force equilibrium at the NODES are

$$\begin{pmatrix} (\vec{X}_A - \vec{X}_B) & \vec{0} & (\vec{X}_A - \vec{X}_C) \\ (\vec{X}_B - \vec{X}_A) & (\vec{X}_B - \vec{X}_C) & \vec{0} \\ \vec{0} & (\vec{X}_C - \vec{X}_B) & (\vec{X}_C - \vec{X}_A) \end{pmatrix} \begin{pmatrix} \tau_a \\ \tau_b \\ \tau_c \end{pmatrix} = \begin{pmatrix} \vec{F}_A \\ \vec{F}_B \\ \vec{F}_C \end{pmatrix}$$

, or

$$\underbrace{\begin{pmatrix} -2 & 0 & -3 \\ -1 & 0 & 0 \\ 2 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \tau_a \\ \tau_b \\ \tau_c \end{pmatrix}}_{\vec{\tau}} = \underbrace{\begin{pmatrix} 0 \\ F \\ 0 \\ -W \\ 0 \\ G \end{pmatrix}}_{\vec{F}}$$

where $\tau = T/L$

T = tension

L = length

for each beam

Here $L_a = \sqrt{5}$, $L_b = \sqrt{2}$, $L_c = 3$

It saves a lot of writing to solve for τ's FIRST, then find T's later!

Before we solve the given system, we must make sure that a solution EXISTS, i.e.

$$\vec{l}^{\,T}\vec{F} = 0 \quad \text{for each null vector } \vec{l} \text{ of } A^T$$

So I input augmented matrix for $A^T\vec{x} = \vec{0}$ into RYCHLIKS applet, and row-reduce it:

$$\left(\begin{array}{cccccc|c} -2 & -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & -1 & 0 \\ -3 & 0 & 0 & 0 & 3 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -3 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{array}\right)$$

(using REDUCED Echelon form!)

So if variables are r, s, t, u, v, w then u, v, w free

$r = v$, $s = 3u - 2w$, $t = u + v - w$

solution

$$\vec{x} = \begin{pmatrix} v \\ 3u-2w \\ u+v-w \\ u \\ v \\ w \end{pmatrix} = u\underbrace{\begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{=\vec{l}^{(1)}} + v\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{=\vec{l}^{(2)}} + w\underbrace{\begin{pmatrix} 0 \\ -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\vec{l}^{(3)}}$$

NOTE: these $\vec{l}$'s look different from the ones I showed in class, BUT they are actually just combinations of each other.

SO $\vec{\ell}^{(1)T}\vec{F} = 0 \Rightarrow 3F - W = 0 \Rightarrow F = W/3$

So if $W = 600 \Rightarrow \boxed{F = 200}$

$\vec{\ell}^{(2)T}\vec{F} = 0$ implies $0+0+0+0+0+0=0$ is AUTOMATICALLY true!

$\vec{\ell}^{(3)T}\vec{F} = 0 \Rightarrow -2F + G = 0 \Rightarrow G = 2F$

if $F = 200$ $\boxed{G = 400}$

These values of F & G guarantee a SOLUTION EXISTS for τ_a, τ_b, τ_c. So NOW we input ORIGINAL SYSTEM into RYCHLIK'S applet

$$\left(\begin{array}{ccc|c} -2 & 0 & -3 & 0 \\ -1 & 0 & 0 & 200 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 0 & -600 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 0 & 400 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -200 \\ 0 & 1 & 0 & -400 \\ 0 & 0 & 1 & 400/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

SURE ENOUGH, the compatibility conditions are satisfied: every zero row in matrix part has 0 on RHS.

AND $\tau_c = 400/3$ (Beam c is in TENSION)

$\tau_b = -400$ (Beams a, b are in COMPRESSION)

$\tau_a = -200$

Therefore the ACTUAL Tensions are

$T_a = \tau_a L_a = -200\sqrt{5} \approx -450$ pounds

$T_b = \tau_b L_b = -400\sqrt{2} \approx -720$ pounds

$T_c = \tau_c L_c = +\frac{400}{3}(3) = 400$ pounds

if $W = 600$ pounds

☆ NOTE: $F = 200$ pounds & $G = 400$ pounds are proportion of total weight supported by the two walls.
ALSO useful quantities to know!

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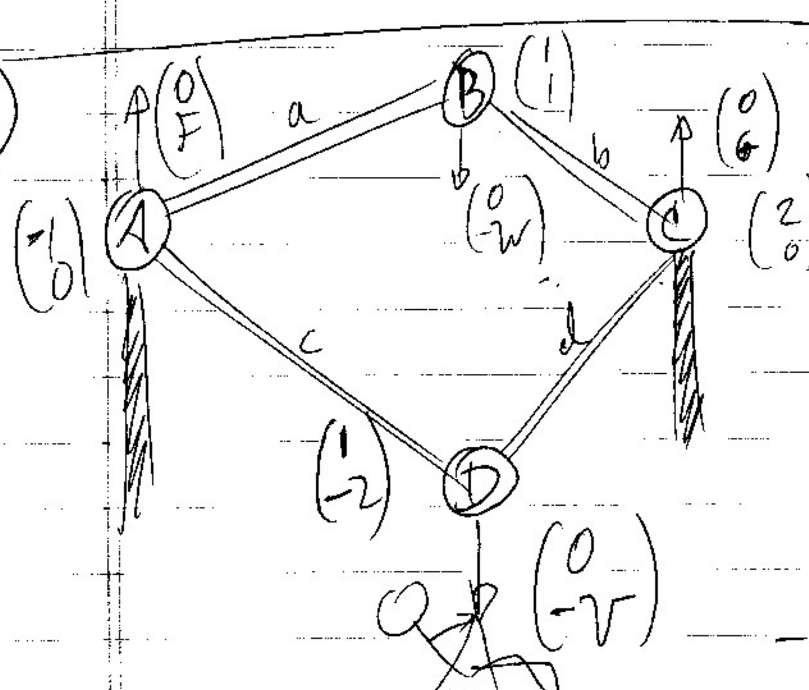
(2)



We EXPECT in absence of crossbeam, the structure might collapse, UNLESS the weight V exactly counter-acts the weight W

[I.e. W tends to push A & C apart; V to pull A & C together. If these balance, don't need the crossbeam.]

THE SAME PROCESS starts with $A^T\vec{x} = \vec{0}$, i.e.

$$
A^T \to \left(\begin{array}{cccccccc|c}
-2 & -1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 0 \\
-2 & 2 & 0 & 0 & 0 & 0 & 2 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 & 0
\end{array}\right)
$$

which reduces to

$$
\begin{array}{c}
\left(\begin{array}{cccccccc|c}
1 & 0 & 0 & -1 & 0 & 2 & -1 & -1 & 0 \\
0 & 1 & 0 & -1 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 1 & -1 & 0 & 3 & -1 & -2 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 & 0
\end{array}\right) \\
\begin{array}{cccccccc}
p & q & r & s & t & u & v & w
\end{array}
\end{array}
$$

variables $p \quad q \quad r \quad s \quad t \quad u \quad v \quad w$

$$
\Rightarrow \begin{pmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} s - 2u + v + w \\ s - 2u + 2w \\ s - 3u + v + 2w \\ s \\ -2u + v + 2w \\ u \\ v \\ w \end{pmatrix} = s\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{l}^{(1)}} + u\underbrace{\begin{pmatrix} -2 \\ -2 \\ -3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{l}^{(2)}} + v\underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{l}^{(3)}} + w\underbrace{\begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{l}^{(4)}}
$$

SO SOLUTION EXISTS to $A\vec{\tau} = \vec{F} = \begin{pmatrix} 0 \\ F \\ 0 \\ -W \\ 0 \\ 0 \\ 0 \\ -V \end{pmatrix}$

provided each $\vec{l}^{\,T}\vec{F} = 0$,

$\vec{l}^{(1)T}\vec{F} =$

Ie. $F - W = 0 \Rightarrow \boxed{F = W = 600}$

$\vec{l}^{(2)T}\vec{F} = -2F + G = 0 \Rightarrow \boxed{G = 2F = 1200}$

$\vec{l}^{(3)T}\vec{F} = 0$ automatically.

$\vec{l}^{(4)T}\vec{F} = 2F - V = 0 \Rightarrow \boxed{V = 2F = 1200}$

This is the extra condition for equilibrium to exist.

Then can solve for ~~those~~ τ's using WYCALC :

$$A\vec{\tau} = \vec{F} : \left(\begin{array}{cccc|c} -2 & 0 & -2 & 0 & 0 \\ -1 & 0 & 2 & 0 & 600 \\ 2 & -1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & -600 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 & 1200 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & -2 & -1200 \end{array}\right) \Rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 0 & 0 & -400 \\ 0 & 0 & 1 & 0 & 200 \\ 0 & 0 & 0 & 1 & 400 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

So $\tau_a = -200$, $T_a = -200\sqrt{5}$ (Q: why are these the same as before?)

$\tau_b = -400$, $T_b = -400\sqrt{2}$

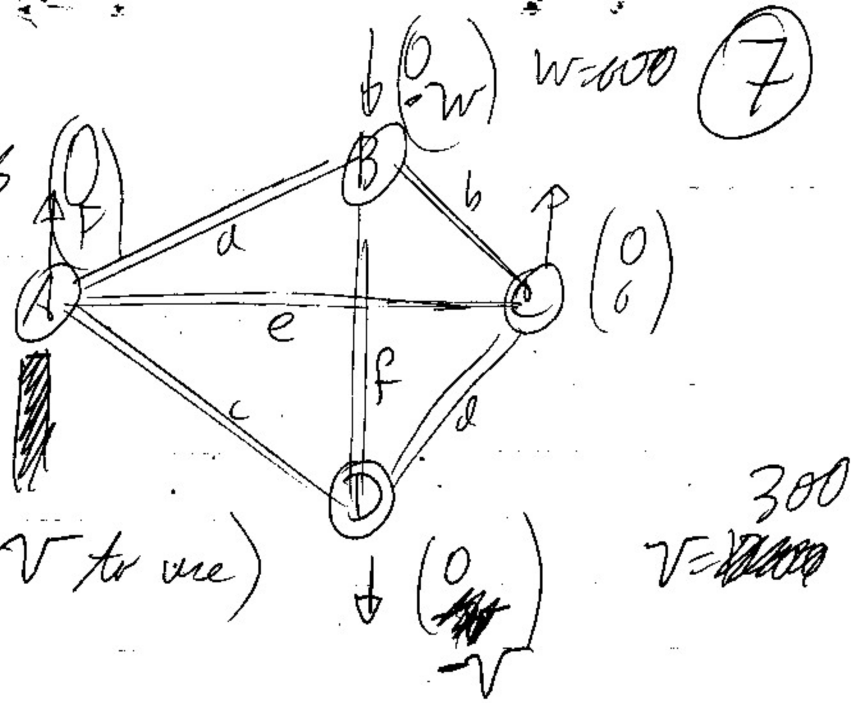
$\tau_c = 200$, $T_c = 200\sqrt{8} = 400\sqrt{2}$

$\tau_d = 400$, $T_d = 400\sqrt{5}$

(3) WITH TWO CROSSBARS

We ~~don't~~ need to balance V versus W!

(I did not specify what V to use)



NOW the $\vec{\tau}$ ~~an~~ satisfy $A\vec{\tau} = \vec{F}$ where

$$A = \begin{pmatrix} -2 & 0 & -2 & 0 & -3 & 0 \\ -1 & 0 & 2 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & -2 & -2 & 0 & -3 \end{pmatrix} \quad \vec{F} = \begin{pmatrix} 0 \\ F \\ 0 \\ -W \\ 0 \\ G \\ 0 \\ -V \end{pmatrix} \quad \vec{\tau} = \begin{pmatrix} \tau_a \\ \tau_b \\ \tau_c \\ \tau_a \end{pmatrix}$$

SIMILARLY $A^T\vec{x} = \vec{0}$ reduces to

$$\begin{array}{c} \left(\begin{array}{cccccccc|c} 1 & 0 & 0 & 0 & 0 & 2 & -1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 & -1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \\ \begin{array}{cccccccc} p & q & r & s & t & u & v & w \end{array} \end{array}$$

variables

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Therefore $\vec{x} = \begin{pmatrix} p \\ q \\ r \\ s \\ t \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -2u+v+2w \\ -2u+3w \\ -3u+v+3w \\ w \\ -2u+v+2w \\ u \\ v \\ w \end{pmatrix} = u \underbrace{\begin{pmatrix} -2 \\ -2 \\ -3 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{l}^{(1)}} + v \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{l}^{(2)}} + w \underbrace{\begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \\ 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{l}^{(3)}}$

Then $\vec{l}^{(1)T}\vec{F} = -2F + G = 0 \Rightarrow \boxed{G = 2F}$

$\vec{l}^{(2)T}\vec{F} = 0$ automatically

$\vec{l}^{(3)T}\vec{F} = 3F - W - V \Rightarrow \boxed{F = \frac{W+V}{3}}$

If $W = 600,\ V = 300$ then

$F = 300,\ G = 600$ and $A\vec{\tau} = \vec{F}$

reduces to $\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 1 & -200 \\ 0 & 1 & 0 & 0 & 0 & 2 & -400 \\ 0 & 0 & 1 & 0 & 0 & 1/2 & 50 \\ 0 & 0 & 0 & 1 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 1 & -1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\Rightarrow$ We now have a FREE VARIABLE τ_f!

so $\tau_a = -200 - \tau_f \qquad \tau_b = -400 - 2\tau_f$

$\tau_c = 50 - \frac{1}{2}\tau_f \qquad \tau_d = 100 + \tau_f$

and $\tau_e = 100 + \tau_f \qquad \tau_f$ is FREE!

[which could be converted to actual tensions using the lengths as before, with $L_e = 3$, $L_f = 3$ also]

The reason for τ_f being free is that the structure is OVERBUILT — we do not NEED beam f at all; the 5 beams a-e can maintain the structure against any external applied force (that satisfies the zero total force & torque). THEN any extra tension T_f or compression exerted by f is balanced by other added forces (proportional to T_f) in the other beams.

MIN LENGTH SOLUTIONS

I forget which I assigned! Lets do 1.8.2 (d) which definitely has interesting behavior:

$$2x - 6y + 4z = 2 \qquad \left(\begin{array}{ccc|c} 2 & -6 & 4 & 2 \\ -1 & 3 & -2 & -1 \end{array}\right) \xrightarrow{l_{21} = -1/2}$$
$$-x + 3y - 2z = -1$$

$$\left(\begin{array}{ccc|c} 2 & -6 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow y, z \text{ both free!}$$

$$x = 1 + 3y - 2z$$

solution $\vec{x} = \begin{pmatrix} 1+3y-2z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$$\text{Length}^2 = \|\vec{x}\|_2^2 = (1+3y-2z)^2 + y^2 + z^2$$

$$L^2 = = 1 + 9y^2 + 4z^2 - 12yz + 6y - 4z + y^2 + z^2$$
$$= 10y^2 + 5z^2 - 12yz + 6y - 4z + 1$$

To minimize, set $\frac{\partial L^2}{\partial y} = \partial L^2 / \partial z = 0$ & solve for y & z.

$$\Rightarrow \cancel{18}\,20y - 12z + 6 = 0$$
$$-12y + 10z - 4 = 0$$

Lets divide everything by 2!

$$\Rightarrow \left(\begin{array}{cc|c} 10 & -6 & -3 \\ -6 & 5 & 2 \end{array}\right) \xrightarrow[= -3/5]{l_{21} = -6/10} \left(\begin{array}{cc|c} 10 & -6 & -3 \\ 0 & 7/5 & 1/5 \end{array}\right)$$

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So $\boxed{z = 1/7}$, $10y - 6/7 = -21/7$

$10y = -15/7$ $\quad \boxed{y = -3/14}$

and $\boxed{x = 1 - 9/14 - 2/7 = 1/14}$ $\quad \vec{x}_{ML} = \begin{pmatrix} 1/14 \\ -3/14 \\ 1/7 \end{pmatrix}$

ALTERNATIVE METHOD: Solve $AA^T \vec{u} = \vec{b}$ for $\vec{u}$,

then $\vec{x}_{ML} = A^T\vec{u}$. $\quad$ Let $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

Here $AA^T = \begin{pmatrix} 2 & -6 & 4 \\ -1 & 3 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -6 & 3 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} 56 & -28 \\ -28 & 14 \end{pmatrix}$

$\Rightarrow \vec{u}$ problem is $\left(\begin{array}{cc|c} 56 & -28 & 2 \\ -28 & 14 & -1 \end{array}\right) \xrightarrow{L_{21} = -1/2} \left(\begin{array}{cc|c} 56 & -28 & 2 \\ 0 & 0 & 0 \end{array}\right)$

$\Rightarrow$ v free, $u = \frac{1}{28} + \frac{1}{2}v \Rightarrow \vec{u} = \begin{pmatrix} 1/28 \\ 0 \end{pmatrix} + v\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

(Note INFINITE NUMBER of $\vec{u}$ solutions!)

$\vec{x}_{ML} = A^T\vec{u} = A^T\begin{pmatrix} 1/28 \\ 0 \end{pmatrix} + vA^T\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 2 & -1 \\ -6 & 3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1/28 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/14 \\ -3/14 \\ 1/7 \end{pmatrix}$ $\quad \rightarrow v \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$!

$\vec{x}_{ML} = \begin{pmatrix} 1/14 \\ -3/14 \\ 1/7 \end{pmatrix}$

$\Rightarrow$ it is UNIQUE!