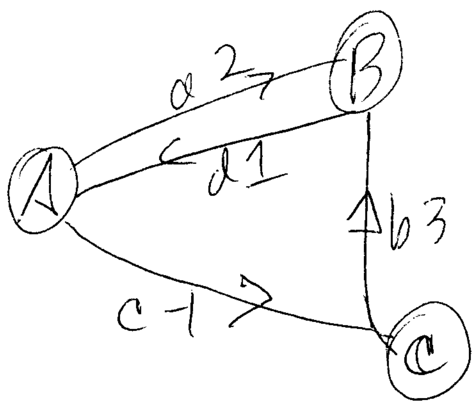
MATH 410 Network problems:



Currency exchange rates:

$1\ \$C = (2^3 = 8)\ \$B$

$1\ \$A = (2^2 = 4)\ \$B$

etc

1 Using edges a, b only $\quad A \xrightarrow{a2} B \xleftarrow{b3} C$

$\Rightarrow \quad x_B - x_A = 2, \quad x_B - x_C = 3 \qquad \left(\begin{array}{ccc|c} -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right)$

is already in Echelon form

$\Rightarrow$ $\underline{x_C \text{ free}}$, $x_B = x_C + 3$, $x_A = x_B - 2 = x_C + 1$

$\Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + x_C \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{k} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is $\underline{\text{null vector}}$

$\vec{x}^{(p)} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$ gives ranking: B best, then A, then C

CURRENCY ~~[illegible]~~ $2^0\ \$C = 2^3\ \$B = 2^1\ \$A$

$1\ \$C = 8\ \$B = 2\ \$A$

ALL CONSISTENT; NO ARBITRAGE OPPORTUNITIES.

2 Adding edge c $\Rightarrow X_C - X_A = -1$

$$\Rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ -1 & 0 & 1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -1 & 1 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

3rd row $\Rightarrow$ NO PROBLEM! Same solution as in problem 1

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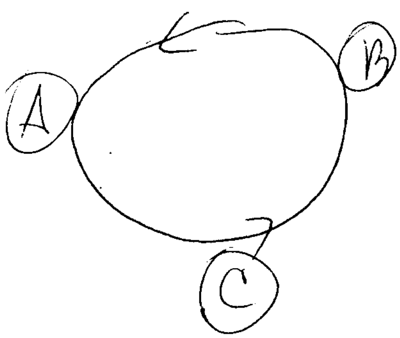
3 Add edge d $\Rightarrow X_A - X_B = 1$ is INCONSISTENT with previous edges! (Since $X_A - X_B = (X_C+1) - (X_C+3)$
$= -2$ according to previous solution)

$\Rightarrow$ we expect $\vec{l}^T \vec{b} \neq 0$

for at least one ~~an~~ basic loop vector $\vec{l}$.

By looking at graph, we can identify TWO Basic loops:

$$\vec{\ell}^{(1)} = \begin{array}{c} a \\ b \\ c \\ d \end{array}\begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{\ell}^{(2)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$



Here $\vec{b} = \begin{pmatrix} 2 \\ 3 \\ -1 \\ 1 \end{pmatrix}$

So $\vec{\ell}^{(1)T}\vec{b} = -2+3-1+0 = 0$ No inconsistency

But $\vec{\ell}^{(2)T}\vec{b} = 2+0+0+1 = 3 \neq 0$!

<u>SO</u> we CAN make money by exchanging around loop $\vec{\ell}^{(2)}$, and the FACTOR by which you increase 1\$A is $2^{(\vec{\ell}^{(2)T}\vec{b})} = 2^3 = 8.$

To rank teams APPROXIMATELY, find LEAST SQUARES approx. solution $\vec{x}_{LS}$ satisfying

$$(A^TA)\vec{X}_{LS} = A^T\vec{b}, \quad \text{where } A = \begin{array}{c|ccc} & A & B & C \\ \hline a & -1 & 1 & 0 \\ b & 0 & 1 & -1 \\ c & -1 & 0 & +1 \\ d & 1 & -1 & 0 \end{array}$$

$$\Rightarrow A^TA = \begin{pmatrix} 3 & -2 & -1 \\ -2 & 3 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad A^T\vec{b} = \begin{pmatrix} 0 \\ 4 \\ -4 \end{pmatrix}$$

so

$$\left(\begin{array}{ccc|c} 3 & -2 & -1 & 0 \\ -2 & 3 & -1 & 4 \\ -1 & -1 & 2 & -4 \end{array}\right) \xrightarrow[\text{computer}]{} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 8/5 \\ 0 & 1 & -1 & 12/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

<u>So</u>

x_c free

$$\vec{X}_{LS} = \begin{pmatrix} x_c + 8/5 \\ x_c + 12/5 \\ x_c \end{pmatrix} = \begin{pmatrix} 8/5 \\ 12/5 \\ 0 \end{pmatrix} + x_c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \leftarrow K$$

(e) So again B best, A next, C last.

BUT different relative strengths!

<u>NOT</u> in HW! But interesting:

<u>OBSERVE</u> that $\vec{X}_{LS}$ has ∞ solutions!

What is Min length $\vec{X}_{LS}$? $= \vec{X}_{LSML}$

Since we already have $\vec{X}_{LS} = \begin{pmatrix} 8/5 + z \\ 12/5 + z \\ 0 + z \end{pmatrix}$ using z for X_C

$$\|\vec{X}_{LS}\|_2^2 = \left(z^2 + \frac{16}{5}z + \frac{64}{25}\right) + \left(z^2 + \frac{24}{5}z + \frac{144}{25}\right) + z^2$$

$$= 3z^2 + 8z + \frac{208}{25}$$

$$\frac{d}{dz}\|\vec{X}_{LS}\|_2^2 = 6z + 8$$

$\stackrel{\text{set}}{=} 0$ when $z = -4/3$

$$\Rightarrow \boxed{\vec{X}_{LSML} = \begin{pmatrix} 8/5 - 4/3 \\ 12/5 - 4/3 \\ -4/3 \end{pmatrix} = \begin{pmatrix} 4/15 \\ 16/15 \\ -4/3 \end{pmatrix}}$$

STILL same information; just more objective — looking.

☆ Other note: Least squares approx. is never used in currency situations! — Prof. N.